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The Riesz Representation Operator on the Dual of $C[0;1]$ is Computable

Tahereh Jafarikhah
(University of Tarbiat Modares, Tehran, Iran
t.jafarikhah@modares.ac.ir)

Klaus Weihrauch
(University of Hagen, Hagen, Germany
Klaus.Weihrauch@FernUni-Hagen.de)

Abstract: By the Riesz representation theorem, for every linear functional $F : C[0;1] \to \mathbb{R}$ there is a function $g : [0;1] \to \mathbb{R}$ of bounded variation such that

$$F(h) = \int h \,\mathrm{d}g \quad (h \in C[0;1]) \,.$$

A computable version is proved in [Lu and Weihrauch(2007)]: a function g can be computed from F and its norm, and F can be computed from g and an upper bound of its total variation. In this article we present a much more transparent proof. We first give a new proof of the classical theorem from which we then can derive the computable version easily. As in [Lu and Weihrauch(2007)] we use the framework of TTE, the representation approach for computable analysis, which allows to define natural concepts of computability for the operators under consideration.

Key Words: computable analysis, Riesz representation theorem
Category: F.0, F.1.1

1 Introduction

The Riesz representation theorem for continuous functionals on $C[0;1]$, the Banach space of continuous functions $h : [0;1] \to \mathbb{R}$ endowed with the supremum norm, can be stated as follows
[Goffman and Pedrick(1965), Heuser(2006)]:

Theorem 1 (Riesz representation theorem). *For every continuous linear operator $F : C[0;1] \to \mathbb{R}$ there is a function $g : [0;1] \to \mathbb{R}$ of bounded variation such that*

$$F(h) = \int h \,\mathrm{d}g \quad (h \in C[0;1])$$

and

$$V(g) = \|F\| \,.$$

Here, $\int h \,\mathrm{d}g$ is the Riemann-Stieltjes integral [Schechter(1997)]. The reversal of this theorem is almost trivial: the operator $h \mapsto \int h \,\mathrm{d}g$ is continuous and linear.

A computable version of the Riesz representation theorem has been proved in [Lu and Weihrauch(2007)]: a function g can be computed from F and its norm, and F can be computed from g and an upper bound of its total variation. This proof, however, is complicated and partly intransparent. In this article we present a simpler and much more transparent proof which starts with a new proof of the classical theorem from which the computable version can be derived easily.

The classical Riesz representation theorem can be proved as follows [Goffman and Pedrick(1965), Heuser(2006)]: By the Hahn-Banach theorem, the operator F has a continuous extension $\overline{F}$ to the Banach space $B[0;1]$ of bounded functions $h : [0;1] \to \mathbb{R}$ such that $\|F\| = \|\overline{F}\|$. Then define g by $g(x) := \overline{F}(\chi_{[0;x]})$, where $\chi_{[0;x]}$ is the characteristic function of $[0;x]$. In our proof, from F and $\|F\|$ we define a dense set of points x in which g will be continuous. For these points x we can compute F to $\chi_{[0;x]}$, then we define $g(x) := \overline{F}(\chi_{[0;x]})$.

In Section 2 we extend the definition of the Variation and the Riemann-Stieltjes integral to partial functions $g :\subseteq [0;1] \to \mathbb{R}$ the domains of which are dense in the unit interval. We observe that $\int h\mathrm{d}g$ can be defined already from any restriction of g to a countable dense subset of it domain.

In Section 3 we introduce the set PC_F of the points x which do not contribute to $\|F\|$ and define $F(\chi_{[0;x]})$ as the limit of $F(h_i)$ where $(h_i)_i$ is a sequence of continuous functons "converging" to $\chi_{[0;x]}$. We prove that g_F is continuous with no continuous proper extension, and that its total variation is $\|F\|$. Furthermore, $F(h) = \int h\mathrm{d}g_F$ for all continuous functions $f : [0;1] \to \mathbb{R}$.

In Section 4 we shortly summarize the computability concepts used in the following. In particular we define our representation of the functions with countable dense domain and finite variation.

Finally, in Section 5 we prove that from F and $\|F\|$ a restriction g of g_F can be *computed* (a function of bounded variation representing F), and that F can be computed from g and a upper bound of $\mathrm{Var}(g)$.

2 The Riemann-Stieltjes integral

We recall the definition of the Riemann-Stieltjes integral. We study only the special case of functions on the unit interval $[0;1]$. Results for arbitrary intervals $[a;b]$ can be derived easily from the special case. In our context it seems to be appropriate to generalize the definitions to partial functions $g :\subseteq [0;1] \to \mathbb{R}$ of bounded variation.

A *partition* of the real interval $[0;1]$ is a sequence $Z = (x_0, x_1, \ldots, x_n)$, $n \geq 1$, of real numbers such that $0 = x_0 < x_1 \ldots < x_n = 1$. The partition Z has *precision* k, if $x_i - x_{i-1} < 2^{-k}$ for $1 \leq i \leq n$. A partition $Z' = (x'_0, x'_1, \ldots, x'_m)$, is finer than Z, if $\{x_0, x_1, \ldots, x_n\} \subseteq \{x'_0, x'_1, \ldots, x'_m\}$. Z is a partition for $g :\subseteq [0:1] \to \mathbb{R}$ if $\{x_0, x_1, \ldots, x_n\} \subseteq \mathrm{dom}(g)$. For a partition Z for g define

$$S(g,Z) := \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|. \tag{1}$$

The *variation* of g is defined by

$$V(g) := \sup\{S(g,Z) | Z \text{ is a partition for } g\}. \tag{2}$$

The function g is of *bounded variation* if its variation $V(g)$ is finite.

Definition 2. Let BV be the set of (partial) functions $g :\subseteq [0;1] \to \mathbb{R}$ of bounded variation such that $\{0,1\}\subseteq\mathrm{dom}(g)$ and $\mathrm{dom}(g)$ is dense in $[0;1]$.

The relation to the usual definitions with total functions g is given by the following lemma.

Lemma 3.

1. *Let $g, g' \in \mathrm{BV}$ such that g is a restriction of g'. Then $V(g) \le V(g')$.*
2. *For every function $g \in BV$ the extension $\overline{g} : [0;1] \to \mathbb{R}$ defined by*

$$\overline{g}(x) := \lim_{y\in\mathrm{dom}(g),\ y\nearrow x} g(y) \qquad \text{for } x \notin \mathrm{dom}(g) \tag{3}$$

is of bounded variation such that $V(g) = V(\overline{g})$.

Proof: (1) Obvious.

(2) Suppose this limit from below does not exist. Then there is an increasing sequence $(y_i)_i$ converging to x such that the sequence $(g(y_i))_i$ does not converge, hence there is some $\varepsilon > 0$ such that $(\forall i)(\exists j > i)\, |g(y_i) - g(y_j)| > \varepsilon$. Therefore, for every n there is some partition $Z_n = (0, y_{i_0}, y_{i_1}, \ldots, y_{i_n}, 1)$ for g such that $S(g, Z_n) > n \cdot \varepsilon$. But g is of bounded variation, hence $\overline{g}(x)$ exists.

Since $\mathrm{dom}(g)\subseteq\mathrm{dom}(\overline{g})$, $V(g) \le V(\overline{g})$. On the other hand suppose $X := (0 = x_1, x_2, \ldots, x_n = 1)$ is a partition for $\overline{g}$ and let $\varepsilon > 0$. For $1 \le i \le n$ there are $y_i \in \mathrm{dom}(g)$ such that $x_{i-1} < y_i < x_i$ and $|g(y_i) - \overline{g}(x_i)| < \varepsilon/(2n)$, hence for $Y := (0, y_1, y_2, \ldots, y_n, 1)$, $|S(\overline{g}, X) - S(g, Y)| < \varepsilon$. Therefore, $V(\overline{g}) \le V(g)$. $\Box$

On the space $C[0;1]$ of continuous functions $h : [0;1] \to \mathbb{R}$ the norm is defined by $\|h\| := \sup_{x\in[0;1]} |h(x)|$. On the space $C'[0;1]$ of the linear continuous operators $F : C[0;1] \to \mathbb{R}$ the norm is defined by $\|F\| := \sup_{\|h\|\le 1} |F(h)|$.

In the following let $h : [0;1] \to \mathbb{R}$ be a continuous function and let $g \in \mathrm{BV}$. For any partition $Z = (x_0, x_1, \ldots, x_n)$ of $[0;1]$ for g define

$$S(g,h,Z) := \sum_{i=1}^{n} h(x_i)(g(x_i) - g(x_{i-1})). \tag{4}$$

Since h is continuous and its domain is compact, it has a (uniform) *modulus of continuity*, i.e., a function $m : \mathbb{N} \to \mathbb{N}$ such that $|h(x) - h(y)| \leq 2^{-k}$ if $|x - y| \leq 2^{-m(k)}$. We may assume that the function m is non-decreasing.

Lemma 4 [Lu and Weihrauch(2007)]. *Let $h : [0; 1] \to \mathbb{R}$ be a continuous function with modulus of continuity $m : \mathbb{N} \to \mathbb{N}$ and let $g \in \mathrm{BV}$. Then there is a unique number $I \in \mathbb{R}$ such that*

$$|I - S(g, h, Z)| \leq 2^{-k} V(g)$$

for every partition Z for g with precision $m(k + 1)$.

A proof is given in [Lu and Weihrauch(2007)]. A revised proof is given in the appendix.

Definition 5. The number I from Lemma 4 is called the *Riemann-Stieltjes integral* and is denoted by $\int h \,\mathrm{d}g$.

Notice that by Lemma 4 the integral $\int f \,\mathrm{d}g$ is determined already by the values of the function g on an arbitrary set X that is dense in $\mathrm{dom}(g)$, since there are partitions of arbitrary precision that contain of points only from the set X.

Corollary 6. *Let $g, g' \in \mathrm{BV}$. Suppose $A \subseteq \mathrm{dom}(g) \cap \mathrm{dom}(g')$ is dense in $[0; 1]$ such that $\{0, 1\} \subseteq A$ and $(\forall x \in A) g(x) = g'(x)$. Then $\int h \,\mathrm{d}g = h \,\mathrm{d}g'$ for every $h \in C[0; 1]$.*

Proof: Obvious. □

3 Another proof of the classical theorem

In this section we present a proof of the (non-computable) Riesz representation theorem which we will effectivize in Section 5. Let Pg be the (countable) set of of polygon functions $h : [0; 1] \to \mathbb{R}$ with rational vertices and let $\mathrm{RI} := \{(a; b) \mid a, b \in \mathbb{Q},\ 0 \leq a < b \leq 1\}$ be the set of open rational subintervals of $(0; 1)$. By the Weierstraß approximation theorem Pg is dense in $C[0; 1]$. In the following let $F : C[0; 1] \to \mathbb{R}$ be a linear continuous functional.

Definition 7. For $h \in C[0; 1]$, $Y \subseteq [0; 1]$, and $x \in (0; 1)$ define $\mathrm{NZ}(h)$, $\|F\|_Y$ and $\mathrm{PC}_F \subseteq (0; 1)$ as follows:

$$\mathrm{NZ}(h) := \{x \in [0; 1] \mid h(x) \neq 0\}\,, \tag{5}$$

$$\|F\|_Y := \sup\{|F(h)| \mid h \in C[0; 1],\ \|h\| \leq 1,\ \mathrm{NZ}(h) \subseteq Y\}\,, \tag{6}$$

$$x \in \mathrm{PC}_F :\Longleftrightarrow \inf\{\|F\|_J \mid x \in J \in \mathrm{RI}\} = 0\,. \tag{7}$$

$\mathrm{NZ}(h)$ is the non-zero region of the function h, $\|F\|_Y$ is the contribution of the set Y to $\|F\|$, and $x \in \mathrm{PC}_F$ means that the contribution of $x \in (0;1)$ to $\|F\|$ is 0. The points from PC_F will be the points of continuity of the associated function g_F of bounded variation.

Lemma 8. *1.* $\|F\|_Y \le \|F\|_Z$ *if* $Y \subseteq Z$,

2. $\|F\|_{J_1} + \ldots + \|F\|_{J_n} \le \|F\|$ *if the* J_i *are pairwise disjoint.*

3. $|F(h_1)| + \ldots + |F(h_n)| \le \|F\|$ *if* $\|h_i\| \le 1$ *for* $i = 1, \ldots, n$ *and the sets* $\mathrm{NZ}(h_i)$ *are pairwise disjoint.*

Proof: (1) Obvious.

(2) Let $\varepsilon > 0$. For $i = 1, \ldots n$ there is a continuous functions h_i such that $\|h_i\| \le 1$, $\mathrm{NZ}(h_i) \subseteq J_i$ and $|F(h_i)| \ge \|F\|_{J_i} - \varepsilon$. We may assume $F(h_i) \ge 0$ (if $F(h_i) < 0$, replace h_i by $-h_i$). Since the sets $\mathrm{NZ}(h_i)$ are pairwise disjoint, $\|\sum_i h_i\| \le 1$. We obtain

$$\sum_i \|F\|_{J_i} \le n\varepsilon + \sum_i |F(h_i)| = n\varepsilon + \sum_i F(h_i) = n\varepsilon + F(\sum_i h_i) \le n\varepsilon + \|F\| \,.$$

This is true for all $\varepsilon > 0$, hence $\sum_i \|F\|_{J_i} \le \|F\|$.

(3)This follows from (2). □

At most countably many points can have a positive contribution to $\|F\|$.

Lemma 9. *The complement* $(0;1) \setminus \mathrm{PC}_F$ *of* PC_F *is at most countable.*

Proof: For $n \in \mathbb{N}$ let T_n be the set of all $x \in (0;1)$ such that $\inf\{\|F\|_J \mid x \in J\} > 2^{-n}$. Suppose, $\mathrm{card}(T_n) \ge N > 2^n \cdot \|F\|$. Then there are N points $x_1, \ldots, x_N \in T_n$ and pairwise disjoint intervals $J_1, \ldots, J_N$ such that $x_i \in J_i$. Since $\|F\|_{J_i} > 2^{-n}$ for all i, $\sum_i \|F\|_{J_i} > N \cdot 2^{-n} > \|F\|$. But this is false by Lemma 8. Therefore, T_n is finite for every n and $(0;1) \setminus \mathrm{PC}_F = \bigcup_n T_n$ is at most countable. □

We define *slanted step functions* (Figure 2) as approximations of characteristic functions $\chi_{[0;x]}$.

Definition 10. For $I = (a;b) \in \mathrm{RI}$ let $s_I \in \mathrm{Pg}$, the *slanted step function* at I, be the polygon function whose graph has the vertices $(0,1)$, $(a,1)$, $(b,0)$, and $(1,0)$.

Suppose $J, K \subseteq L$. Then $\mathrm{NZ}(s_J - s_K) \subseteq L$ and $\|s_J - s_K\| \le 1$, hence $|F(s_J) - F(s_K)| = |F(s_J - s_K)| \le \|F\|_L$, therefore

$$|F(s_J) - F(s_K)| \le \|F\|_L \text{ if } J, K \subseteq L \,. \tag{8}$$

In the classical proof (Section 1) $g(x)$ can be defined as $\overline{F}(\chi_{[0;x]})$, where $\overline{F}$ is the Hahn-Banach extension of F to the bounded real functions. We replace this definition as follows considering only points of continuity:

Definition 11. Define a function $g_F :\subseteq \mathbb{R} \to \mathbb{R}$ as follows: $\mathrm{dom}(g_F) := \{0,1\} \cup \mathrm{PC}_F$, $g(0) := 0$, $g(1) := F(1)$. For $x \in \mathrm{PC}_F$ let $(J_n)_{n\in\mathbb{N}}$ be a sequence of rational intervals such that $x \in J_{n+1} \subseteq J_n$ and $\lim_{n\to\infty} \mathrm{length}(J_n) = 0$. Then let $g_F(x) := \lim_{n\to\infty} F(s_{J_n})$.

Since $x \in \mathrm{PC}_F$, $\lim_{n\to\infty} \|F\|_{J_n} = 0$ by monotonicity in J of $\|F\|_J$. We show that $g_F(x)$ exists and does not depend on the specific sequence$(J_n)_{n\in\mathbb{N}}$.

Lemma 12. *The function g_F is well-defined.*

Proof: For every $\varepsilon > 0$ there is some n such that $\|F\|_{J_n} < \varepsilon$. By (8) for $k > n$, $|F(s_{J_n}) - F(s_{J_k})| \leq \|F\|_{J_n} < \varepsilon$, hence $\lim_{n\to\infty} F(s_{J_n})$ exists.

Let $(L_n)_{n\in\mathbb{N}}$ be another sequence of rational intervals such that $x \in L_{n+1} \subseteq L_n$ and $\lim_{n\to\infty} \|F\|_{L_n} = 0$. Then $\lim_{n\to\infty} F(s_{L_n})$ exists accordingly. Let $K_n := J_n \cap L_n$. By (8), $|F(s_{J_n}) - F(s_{K_n})| \leq \|F\|_{J_n}$ and $|F(s_{L_n}) - F(s_{K_n})| \leq \|F\|_{L_n}$, hence $|F(s_{J_n}) - F(s_{L_n})| \leq \|F\|_{J_n} + \|F\|_{L_n}$. Therefore,
$\lim_n |F(s_{J_n}) - F(s_{L_n})| = 0$ and finally $\lim_n F(s_{J_n}) = \lim_n F(s_{L_n})$. □

Lemma 13. *Suppose $J, K, L \in \mathrm{RI}$, $J, K \subseteq L$ and $x, y \in \mathrm{PC}_F \cap L$. Then*

$$|F(s_J) - F(s_K)| \leq \|F\|_L\,, \tag{9}$$

$$|F(s_J) - g_F(y)| \leq \|F\|_L\,, \tag{10}$$

$$|g_F(x) - g_F(y)| \leq \|F\|_L\,. \tag{11}$$

Proof:

(9): By (8).

(10): For every $\varepsilon > 0$ there is some $K \subseteq L$ such that $y \in K$ and $|F(s_K) - g_F(y)| \leq \varepsilon$. Then by (9), $|F(s_J) - g_F(y)| \leq |F(s_J) - F(s_K)| + |F(s_K) - g_F(y)| \leq \|F\|_L + \varepsilon$. Therefore $|F(s_J) - g_F(y)| \leq \|F\|_L$.

(11): For every $\varepsilon > 0$ there is some $J \subseteq L$ such that $x \in J$ and $|F(s_J) - g_F(x)| \leq \varepsilon$. Then by (10), $|g_F(x) - g_F(y)| \leq |g_F(x) - F(s_J)| + |F(s_J) - g_F(y)| \leq \|F\|_L + \varepsilon$. Therefore $|g_F(x) - g_F(y)| \leq \|F\|_L$. □

We will prove some further properties of the function g_F. In the following, $\lim_{y\nearrow x} g_F(y)$ abbreviates $\lim_{y\in\mathrm{dom}(g_F),\ y\nearrow x} g_F(y)$ and $\lim_{y\searrow x} g_F(y)$ abbreviates $\lim_{y\in\mathrm{dom}(g_F),\ y\searrow x} g_F(y)$.

Lemma 14. *For all $x \in (0;1)$,*

1. $\lim_{y\nearrow x} g_F(y)$ *and* $\lim_{y\searrow x} g_F(y)$ *exist,*

2. $|\lim_{y\nearrow x} g_F(y) - \lim_{y\searrow x} g_F(y)| = \inf_{x\in J} \|F\|_J\,.$

Proof:

(1) Suppose that $\lim_{y\nearrow x} g_F(y)$ does not exist. Then there is an increasing sequence $(y_i)_i$ from PC_F converging to x such that the sequence $(g_F(y_i))_i$ does not converge, hence there is some $\varepsilon > 0$ such that $(\forall N)(\exists i,j > N)\,|g_F(y_i)-g_F(y_j)| > \varepsilon$. Therefore, for every N we can find $y_{i_0} < \ldots < y_{i_{2N}}$ from the sequence $(y_i)_i$ such that $|g_F(y_{i_{2k}}) - g_F(y_{i_{2k-1}})| > \varepsilon$, for $1 \le k \le N$. Hence there are pairwise disjoint rational intervals $J_1, J_2, \ldots, J_N$ such that $y_{i_{2k-1}}, y_{i_{2k}} \in J_k$ for $1 \le k \le N$. Then by (11), $||F||_{J_k} > \varepsilon$ for each $1 \le k \le N$. By Lemma 8, $\|F\| \ge \sum_{k=1}^{N} \|F\|_{J_k} > N \cdot \varepsilon$. Since this is true for all numbers N, $||F||$ is unbounded. Contradiction.

(2) Let $a = \inf_{x\in J} \|F\|_J$ and $\delta > 0$. There is some $J \in \mathrm{RI}$ such that

$$x \in J \quad \text{and} \quad |\,\|F\|_J - a| < \delta\,. \tag{12}$$

"$\le$": By (11) and (12) for $y_1, y_2 \in J\cap\mathrm{PC}_F$, $|g_F(y_1)-g_F(y_2)| \le \|F\|_J < a+\delta$, hence $|\lim_{y\nearrow x} g_F(y) - \lim_{y\searrow x} g_F(y)| \le a+\delta$. Since this is true for all $\delta > 0$, "$\le$" is true.

"$\ge$": An example of the functions, intervals etc. defined in the following is shown in Figure 1. There is a rational polygon h such that

$$\mathrm{NZ}(h)\subseteq J,\ \|h\| \le 1 \quad \text{and} \quad |F(h) - \|F\|_J| < \delta\,.$$

The function h can be chosen such that

$$K\subseteq J; \quad x \in K \text{ and } \ (\forall y \in K)\ h(y) = c \tag{13}$$

for some $K \in \mathrm{RI}$ and some c such that $0 < |c| \le 1$. We may assume $0 < c \le 1$ (if $c < 0$ replace h by $-h$). There are $y_<, y_> \in K \cap \mathrm{PC}_F$, $y_< < x < y_>$ such that

$$|\lim_{y\nearrow x} g_F(y) - g_F(y_<)| < \delta \quad \text{and} \quad |\lim_{y\searrow x} g_F(y) - g_F(y_>)| < \delta\,. \tag{14}$$

There are $L, R \in \mathrm{RI}$ such that $L, R\subseteq K$, $L < x < R$, $y_< \in L$, $y_> \in R$ and

$$\|F\|_L < \delta \quad \text{and} \quad \|F\|_R < \delta\,. \tag{15}$$

Let m_L and m_R be the center of L and R respectively. Let $t_L : [0;1] \to \mathbb{R}$ be the rational polygon whose graph has the vertices $(0,0), (\inf L, 0), (m_L, c), (\sup L, 0), (1,0)$ and let $t_R : [0;1] \to \mathbb{R}$ be the rational polygon whose graph has the vertices $(0,0), (\inf R, 0), (m_R, c), (\sup R, 0), (1,0)$. Then $|F(t_L)| \le \|F\|_L < \delta$ and $|F(t_R)| \le \|F\|_R < \delta$.

Let $h' := h - t_L - t_R$. Then

$$|F(h') - F(h)| = |F(t_L) + F(t_R)| \le 2\delta\,. \tag{16}$$

Let N be the interval $(m_L; m_R)$. Let h_0 be the polygon function whose graph has the vertices $(0,0), (m_L, 0), (\sup L, c), (\inf R, c), (m_R, 0), (1,0)$. Let $\overline{h} := h' - h_0$.

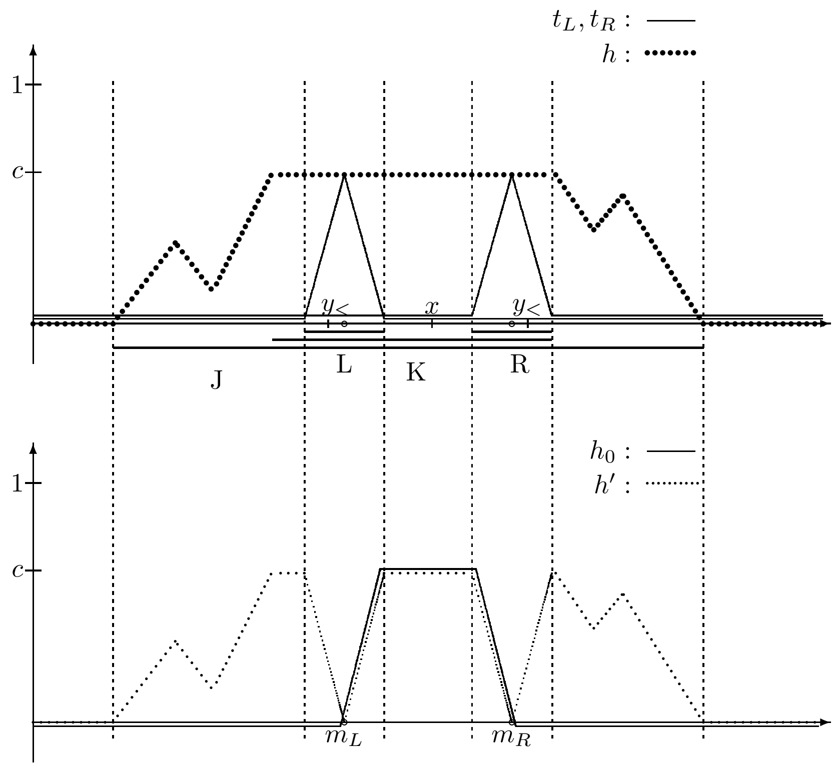

Figure 1: *The functions h,h_0 and h'*

We will show that $|F(\overline{h})|$ is small and $|F(h_0)| \approx a$. There is some rational polygon function h'_0 such that $\|h'_0\| = 1$, $\mathrm{NZ}(h'_0) \subseteq N$ and

$$| \, \|F\|_N - F(h'_0)| < \delta \, . \tag{17}$$

There are $\alpha, \beta \in \{1, -1\}$ such that $|F(h'_0)| + |F(\overline{h})| = F(\alpha h'_0) + F(\beta \overline{h}) = F(\alpha h'_0 + \beta \overline{h})$. Since $\mathrm{NZ}(h'_0) \cap \mathrm{NZ}(\overline{h}) = \emptyset$, $\|\alpha h'_0 + \beta \overline{h}\| \leq 1$, hence $|F(h'_0)| + |F(\overline{h})| \leq \|F\|_J \leq a + \delta$. Since $\|F_N\| \leq |F(h'_0)| + \delta$ and $\|F\|_N \geq a$ because of $x \in N$,

$$|F(h') - F(h_0)| = |F(\overline{h})| \leq a + \delta - |F(h'_0)| \leq a + \delta - \|F\|_N + \delta \leq 2\delta \, .$$

Therefore $F(\overline{h})$ is small. From the above estimations,
$|a| \leq |a - \|F\|_J| + | \, \|F\|_J - F(h)| + |F(h) - F(h')| + |F(h') - F(h_0)| + |F(h_0)| \, ,$
hence $a \leq \delta + \delta + 2\delta + 2\delta + |F(h_0)|$, that is,

$$a \leq 6\delta + |F(h_0)| \, .$$

Therefore, $|F(h_0)|$ is big. By construction, $0 < c = \|h_0\| \leq 1$. Let $\widehat{h} := h_0/c$. Then $a \leq 6\delta + |F(\widehat{h})|$.

Since $\|\widehat{h}\| = 1$, $\widehat{h} = s_T - s_S$ where $S = (m_L; \sup L)$ and $T = (\inf R; m_R)$. By Lemma 13,

$$|g_F(y_<) - F(s_S)| \le \|F\|_K \quad \text{and} \quad |g_F(y_>) - F(s_T)| \le \|F\|_K \,,$$

hence by Lemma 13,

$$\begin{aligned}
a &\le 6\delta + |F(\widehat{h})| \\
&= 6\delta + |F(s_T) - F(s_S)| \\
&\le 6\delta + |F(s_T) - g_F(y_>)| + |g_F(y_>) - \lim_{y \searrow x} g_F(y)| \\
&\quad + |\lim_{y \searrow x} g_F(y) - \lim_{y \nearrow x} g_F(y)| + |\lim_{y \nearrow x} g_F(y) - g_F(y_<)| + |g_F(y_<) - F(s_S)| \\
&\le 6\delta + \|F\|_R + \delta + |\lim_{y \searrow x} g_F(y) - \lim_{y \nearrow x} g_F(y)| + \delta + \|F\|_L \\
&\le |\lim_{y \searrow x} g_F(y) - \lim_{y \nearrow x} g_F(y)| + 10\delta
\end{aligned}$$

Since this is true for all $\delta > 0$, "$\ge$" has been proved. □

Theorem 15.

1. *g_F is continuous on $(0;1) \cap \mathrm{dom}(g_F) = \mathrm{PC}_F$,*
2. *no proper extension g of g_F is continuous on $(0;1) \cap \mathrm{dom}(g)$,*
3. *$\mathrm{Var}(g) = \|F\|$ for every restriction $g \in \mathrm{BV}$ of g_F,*
4. *$\mathrm{Var}(g_F) = \|F\|$.*

Proof: 1. If $x \in \mathrm{PC}_F$ then $\lim_{y \searrow x} g_F(y) = \lim_{y \nearrow x} g_F(y)$ by Lemma 14. Therefore g_F is continuous in x.

2. Let g be an extension of g_F and let g be continuous in $x \in \mathrm{dom}(g)$. Then $\lim_{y \searrow x} g_F(y) = \lim_{y \nearrow x} g_F(y)$, hence $\inf_{x \in J} \|F\|_J = 0$ by Lemma 14, that is, $x \in \mathrm{PC}_F$.

3. **$\mathbf{Var}(\boldsymbol{g}) \le \|\boldsymbol{F}\|$**: Let $X := (x_0, x_1, \ldots, x_n)$ be a partition for g. Let $\varepsilon > 0$. By the definition of g_F for every $0 < i < n$ there is an interval $K_i \in \mathrm{RI}$ such that $x_i \in K_i$, $\sup K_i < \inf K_{i+1}$, $\|F\|_{K_i} < \varepsilon$. Furthermore, for $0 < i < n$ there are intervals $L_i, R_i \in \mathrm{RI}$ such that $L_i, R_i \subseteq K_i$ and $\sup L_i < x_i < \inf R_i$. Figure 2 shows the intervals and some corresponding slanted step functions. By Lemma 8 and Lemma 13,

$$\begin{aligned}
S(g, X) &= |g(x_1)| + \sum_{i=2}^{n-1} |g(x_i) - g(x_{i-1})| + |g(1) - g(x_{n-1})| \\
&\le |F(s_{L_1})| + \varepsilon + \sum_{i=2}^{n-1} (|F(s_{L_i} - s_{R_{i-1}})| + 2\varepsilon) \\
&\quad + |F(1 - s_{R_{n-1}})| + \varepsilon \\
&\le 2n\varepsilon + \|F\| \,.
\end{aligned}$$

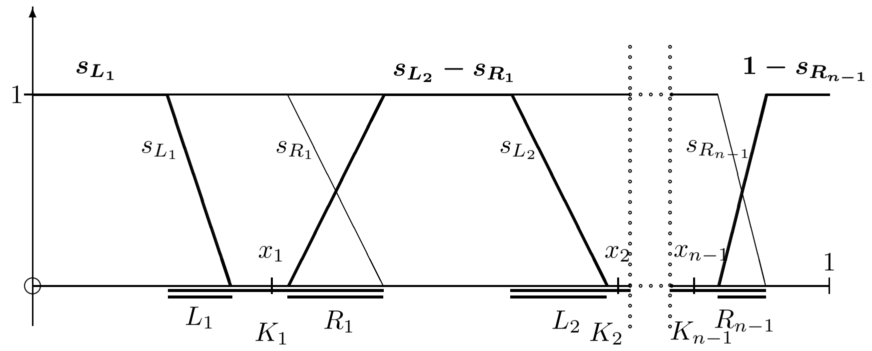

Figure 2: The intervals K_i, L_i, R_i and corresponding slanted step functions.

Since this is true for all $\varepsilon > 0$, $S(g, X) \leq \|F\|$. Since this is true for all partitions X for g, $\mathrm{Var}(g) \leq \|F\|$.

3. $\|\boldsymbol{F}\| \leq \mathbf{Var}(\boldsymbol{g})$: First we show that for every rational polygon function $h_0 \in \mathrm{Pg}$ there are a partition $X = (0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1)$ and intervals K_i, L_i, R_i such that for the function h_2 (see Figure 3), $F(h_0)$ is close to $F(h_2)$ if $(x_i - x_{i-1})$ and $\|F\|_{K_i}$ are sufficiently small for all $1 < i \leq n$. By Lemma 13 $F(h_2)$ can be related to $S(g, X)$ (and to $S(g, h_0, X)$ in the proof of Theorem 16).

Let $h_0 \in \mathrm{Pg}$ and $k \in \mathbb{N}$. Let $m : \mathbb{N} \to \mathbb{N}$ be a modulus of continuity of h_0. Let $n := 2^{m(k)+1} + 1$. Since $\mathrm{dom}(g)$ is dense, there is a partition $X = (0 = x_0, x_1, \ldots, x_{n-1}, x_n = 1)$ for g such that $x_i - x_{i-1} < 2^{-m(k)-1}$. Since all the $x_i \in \mathrm{PC}_F$, for every $0 < i < n$ there are rational intervals K_i, L_i, R_i such that

$$x_i \in K_i, \quad 0 < \inf K_1, \quad \sup K_i < \inf K_{i+1}, \quad \sup K_{n-1} < 1,$$
$$\|F\|_{K_i} < 2^{-k}/n\,,$$
$$\inf L_i = \inf K_i, \quad \sup L_i < x_i < \inf R_i \quad \sup R_i = \sup K_i\,.$$

Figure 3 shows an example of the left end of the unit interval with the function h_0 and the intervals.

For $1 \leq i \leq n$ define

$$c_i \ := \ \max\{h_0(x) \mid \sup R_{i-1} \leq x \leq \inf L_i\}\,,$$

(where $\sup R_0 := 0$ and $\inf L_n := 1$). Define a rational polygon function h_1 by the following sequence of vertices:
$(\sup R_0, c_1), (\inf L_1, c_1), (\sup R_1, c_2), (\inf L_2, c_2), \ldots, (\sup R_{n-1}, c_n), (\inf L_n, c_n)$ (see Figure 3, notice that c_i may be negative).

Suppose $1 \leq i \leq n$ and $\sup R_{i-1} \leq x \leq \inf L_i$. Then $x_{i-1} \leq x \leq x_i$ and $h_1(x) = c_i = h_0(y)$ for some y with $x_{i-1} \leq y \leq x_i$. Then $|x - y| < 2^{-m(k)}$, hence $|h_1(x) - h_0(x)| = |h_0(y) - h_0(x)| < 2^{-k}$.

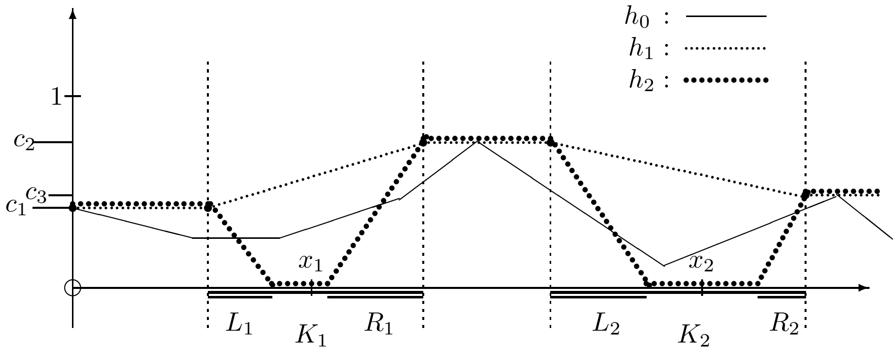

Figure 3: The functions h_0, h_1 and h_2..

Suppose $0 < i < n$ and $x \in K_i$. Then $h_1(x) = h_0(y)$ for some y such that $x_{i-1} < y < x_{i+1}$. Since $x_{i-1} < x < x_{i+1}$, $|x - y| < 2^{-m(k)}$ and hence $|h_1(x) - h_0(x)| = |h_0(y) - h_0(x)| < 2^{-k}$.
Therefore, $\|h_1 - h_0\| < 2^{-k}$ and hence $|F(h_1) - F(h_0)| \le \|F\| \cdot 2^{-k}$.

Let $1 \le i \le n$. Then $c_i = h_0(y)$ for some $x_{i-1} \le y \le x_i$. Since $|x_i - y| < 2^{-m(k)}$, $|h_0(x_i) - c_i| = |h_0(x_i) - h_0(y)| \le 2^{-k}$.

From h_1 we construct a third function h_2 by replacing for every $0 < i < n$ the line segment from $(\inf L_i, c_i)$ to $(\sup R_i, c_{i+1})$ in the graph of h_1 by the polygon $(\inf L_i, c_i)$, $(\sup L_i, 0)$, $(\inf R_i, 0)$, $(\sup R_i, c_{i+1})$ (see Figure 3). Then by Definition 10,

$$h_2 = c_1 s_{L_1} + \sum_{i=2}^{n-1} c_i (s_{L_i} - s_{R_{i-1}}) + c_n (1 - s_{R_{n-1}})\,.$$

For $0 < i < n$ let d_i be the polygon function defined by the sequence of vertices

$$(0,0),\ (\inf L_i, 0),\ (\sup L_i, h_1(\sup L_i)),\ (\inf R_1, h_1(\inf R_1)),\ (\sup R_i, 0),\ (1,0)\,.$$

Then $h_2 = h_1 - \sum_{i=1}^{n-1} d_i$. Since $\mathrm{NZ}(d_i) \subseteq K_i$ and $\|d_i\| \le \|h_0\|$,

$$|F(h_2) - F(h_1)| \le \sum_{i=1}^{n-1} |F(d_i)| \le \sum_{i=1}^{n-1} \|F\|_{K_i} \cdot \|h_0\| \le \|h_0\| \cdot 2^{-k}\,.$$

We prove $\|F\| \le \mathrm{Var}(g)$. There is some $h_0 \in \mathrm{Pg}$ such that $\|h_0\| \le 1$ and $\|F\| \le |F(h_0)| + 2^{-k}$. Since $|c_i| \le 1$ and by Lemma 13,

$$\begin{aligned}
\|F\| &\le |F(h_0 - h_1)| + |F(h_1 - h_2)| + |F(h_2)| + 2^{-k} \\
&\le \|F\| \cdot 2^{-k} + \|h_0\| \cdot 2^{-k} + |F(h_2)| + 2^{-k} \\
&\le |F(s_{L_1})| + \sum_{i=2}^{n-1} |F(s_{L_i} - s_{R_{i-1}})| + |F(1 - s_{R_{n-1}})| \\
&\qquad + (\|F\| + 2) \cdot 2^{-k} \\
&\le |g(x_1)| + 2^{-k}/n + \sum_{i=2}^{n-1} (|g(x_i) - g(x_{i-1})| + 2 \cdot 2^{-k}/n) \\
&\qquad + |g(1) - g(x_{n-1})| + 2^{-k}/n + (\|F\| + 2) \cdot 2^{-k} \\
&\le \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| + 2 \cdot 2^{-k} + (\|F\| + 2) \cdot 2^{-k} \\
&= S(g, X) + (\|F\| + 4) \cdot 2^{-k} \\
&\le \mathrm{Var}(g) + (\|F\| + 4) \cdot 2^{-k} .
\end{aligned}$$

Since this is true for all k, $\|F\| \le \mathrm{Var}(g)$.

4. This follows from 3. □

Theorem 16. *Let* $g \in \mathrm{BV}$ *be a restriction of* g_F. *Then for every* $h \in C[0;1]$, $F(h) = \int h \, \mathrm{d}g$.

Proof: Let $h \in C[0;1]$ and $k \in \mathbb{N}$. There is a function $h_0 \in \mathrm{Pg}$ such that $\|h - h_0\| \le 2^{-k}$. Let $m, n, X, K_i, L_i, R_i, c_i, h_1, h_2$ be the objects introduced in the proof of Theorem 15.3. We prove that $|F(h) - S(g, h, X)|$ is small. By the results that we have already shown,

$$\begin{aligned}
|F(h) - F(h_2)| &\le |F(h) - F(h_0)| + |F(h_0) - F(h_1)| + |F(h_1) - F(h_2)| \\
&\le \|F\| \cdot 2^{-k} + \|F\| \cdot 2^{-k} + \|h_0\| \cdot 2^{-k} \\
&= (2\|F\| + \|h_0\|) \cdot 2^{-k}
\end{aligned}$$

Since $|F(s_{R_i}) + B| \le |g(x_i) + B| + \|F\|_{K_i}$ etc. by Lemma 13, $c_i \le \|h_0\|$, and $|h_0(x_i) - c_i| \le 2^{-k}$,

$$\begin{aligned}
&|F(h_2) - S(g, h_0, X)| \\
\le &\Big| c_1 F(s_{L_1}) + \sum_{i=2}^{n-1} c_i (F(s_{L_i}) - F s_{R_{i-1}})) + c_n (F(1) - F(s_{R_{n-1}})) \\
&\quad - \sum_{i=1}^{n} h_0(x_i)(g(x_i) - g(x_{i-1})) \Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \Big| c_1 g(x_1) + \sum_{i=2}^{n-1} c_i(g(x_i) - g(x_{i-1})) + c_n(g(1) - g(x_{n-1})) \\
&\qquad - \sum_{i=1}^{n} h_0(x_i)(g(x_i) - g(x_{i-1}))\Big| \\
&\qquad + |c_1|\, \|F\|_{K_1} + \sum_{i=2}^{n-1} |c_i|(\|F\|_{K_i} + \|F\|_{K_{i-1}}) + |c_n|\, \|F\|_{K_{n-1}} \\
&\leq \Big| \sum_{i=1}^{n} (c_i - h_0(x_i))(g(x_i) - g(x_{i-1}))\Big| + 2\, \|h_0\| \cdot 2^{-k} \\
&\leq \sum_{i=1}^{n} |c_i - h_0(x_i)| \cdot |g(x_i) - g(x_{i-1})| + \|h_0\| \cdot 2^{-k+1} \\
&\leq 2^{-k} \cdot S(g, X) + \|h_0\| \cdot 2^{-k+1} \\
&\leq 2^{-k} \cdot \mathrm{Var}(g) + \|h_0\| \cdot 2^{-k+1} \\
&= (\|F\| + 2\, \|h_0\|) \cdot 2^{-k}
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
|S(g, h_0, X) - S(g, h, X)| &= |\sum_{i=1}^{n} (h_0(x_i) - h(x_i))(g(x_i) - g(x_{i-1}))| \\
&\leq 2^{-k-} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})| \\
&= 2^{-k} \cdot S(g, X) \\
&\leq 2^{-k} \cdot \mathrm{Var}(g) \\
&= 2^{-k} \cdot \|F\| .
\end{aligned}
$$

Combining these results we obtain

$$
\begin{aligned}
&|F(h) - S(g, h, X)| \\
&\leq |F(h) - F(h_2)| + |F(h_2) - S(g, h_0, X)| + |S(g, h_0, X) - S(g, h, X)| \\
&\leq (2\|F\| + \|h_0\|) \cdot 2^{-k} + (\|F\| + 2\, \|h_0\|) \cdot 2^{-k} + 2^{-k} \cdot \|F\| \\
&\leq (\|F\| + \|h\| + 1) \cdot 2^{-k+2}
\end{aligned}
$$

Since X has precision $m(k)$, $|\int h\, \mathrm{d}g - S(g, h, X)| \leq \mathrm{Var}(g) \cdot 2^{-k+1}$ by Lemma 4. Therefore, $|F(h) - \int h\, \mathrm{d}g| \leq (3\|F\| + 2\|h\| + 2) \cdot 2^{-k+1}$. Since this is true for all k, $F(h) = \int h\, \mathrm{d}g$. □

4 Concepts from Computable Analysis

For studying computability we use the representation approach (TTE) for Computable Analysis [Weihrauch(2000), Brattka et al.(2008)]. Let Σ be a finite alphabet. Computable functions on Σ^* (the set of finite sequences over Σ) and Σ^ω (the set of infinite sequences over Σ) are defined by Turing machines which map sequences to sequences (finite or infinite). On Σ^* and Σ^ω finite or countable tupling will be denoted by $\langle\ \rangle$ [Weihrauch(2000)]. The tupling functions and the projections of their inverses are computable.

In TTE, sequences from Σ^* or Σ^ω are used as "names" of abstract objects such as rational numbers, real numbers, real functions or points of a metric space. We consider computability of multi-functions w.r.t. multi-representations [Weihrauch(2000)], [Brattka et al.(2008)], [Weihrauch(2008), Sections 3,6,8,9].

A representation of a set X is a function $\delta :\subseteq C \to X$ where $C = \Sigma^*$ or $C = \Sigma^\omega$. If $\delta(p) = x$ we call p a δ-*name* of x. If $f : X \rightrightarrows Y$ is a multi-function (on represented sets) then $f(x)$ is the set of $y \in Y$ which are accepted as a result of f applied to x. (Example: $f : \mathbb{R} \rightrightarrows \mathbb{Q}$, $f(x) := \{a \in \mathbb{Q} \mid x < a\}$, we may say: "the multi-function f finds some rational upper bound of x".)

For representations $\gamma :\subseteq Y \to M$ and $\gamma_0 :\subseteq Y_0 \to M_0$, a function $h :\subseteq Y \to Y_0$ is a (γ, γ_0)-realization of a multi-function $f :\subseteq M \rightrightarrows M_0$, iff for all $p \in Y$ and $x \in M$,

$$\gamma(p) = x \in \mathrm{dom}(f) \Longrightarrow \gamma_0 \circ h(p) \in f(x)\,. \tag{18}$$

Fig. 4 illustrates the definition.

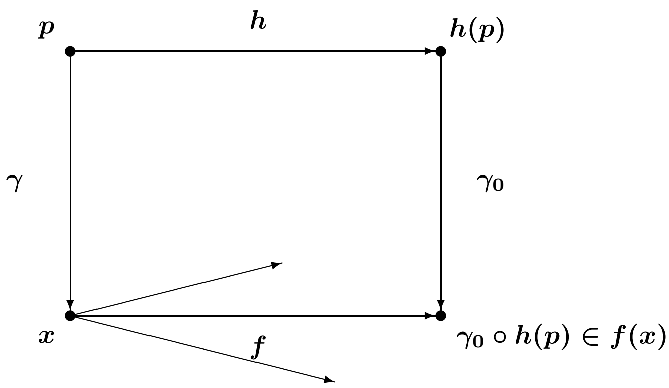

Figure 4: $h(p)$ is a name of some $y \in f(x)$, if p is a name of $x \in \mathrm{dom}(f)$.

The multi-function f is called (γ, γ_0)-computable, if it has a computable (γ, γ_0)-realization and (γ, γ_0)-continuous if it has a continuous realization. The definitions can be generalized straightforwardly to multi-functions $f : M_1 \times \ldots \times M_n \rightrightarrows M_0$ for represented sets M_i.

For two representations $\delta_i :\subseteq \Sigma^\omega \to M_i$ $(i = 1, 2)$ the canonical representation $[\delta_1, \delta_2]$ of the product $M_1 \times M_2$ is defined by

$$[\delta_1, \delta_2]\langle p_1, p_2\rangle = (\delta_1(p_1), \delta(p_2))\,. \tag{19}$$

For two representations $\delta_i {\subseteq} \Sigma^\omega \rightrightarrows M_i$ $(i = 1, 2)$, $\delta_1 \leq \delta_2$ (δ_1 is reducible to δ_2) iff there is a computable function $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that $(\forall\, p \in \mathrm{dom}(\delta_1))\, \delta_1(p) = \delta_2 h(p)$. (If p is a δ_1-name of x then $h(p)$ is a δ_2-name of x.)

We use various canonical notations $\nu :\subseteq \Sigma^* \to X$: $\nu_\mathbb{N}$ for the natural numbers, $\nu_\mathbb{Q}$ for the rational numbers, ν_{Pg} for the polygon functions on $[0;1]$ whose graphs have rational vertices, and ν_I for the set RI open subintervals $(a;b){\subseteq}(0;1)$ with rational endpoints. For functions $m : \mathbb{N} \to \mathbb{N}$ we use the canonical representation $\delta_\mathbb{B} :\subseteq \Sigma^\omega \to \mathbb{B} = \{m \mid m : \mathbb{N} \to \mathbb{N}\}$ defined by $\delta_\mathbb{B}(p) = m$ if $p = 1^{m(0)}01^{m(1)}01^{m(2)}0\ldots$. For the real numbers we use the Cauchy representation $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$, $\rho(p) = x$ if p is (encodes) a sequence $(a_i)_{i\in\mathbb{N}}$ of rational numbers such that for all i, $|x - a_i| \leq 2^{-i}$. By the Weierstraß approximation theorem the countable set of Pg of polygon functions with rational vertices is dense in $C[0;1]$. Therefore, $C[0;1]$ with notation ν_{Pg} of the set Pg is a computable metric space [Weihrauch(2000)] for which we use the Cauchy representation δ_C defined as follows: $\delta_C(p) = h$ if p is (encodes) a sequence $(h_i)_{i\in\mathbb{N}}$ of polygons $h_i \in \mathrm{Pg}$ such that for all i, $\|h - h_i\| \leq 2^{-i}$ [Weihrauch(2000)]. For the space $C(C[0;1], \mathbb{R})$ of the continuous (not necessarily linear) functions $F : C[0;1] \to \mathbb{R}$ we use the canonical representation $[\delta_C \to \rho]$ [Weihrauch(2000), Weihrauch and Grubba(2009)]. It is determined uniquely up to equivalence by (**U**) and (**S**):

- **(U)** the function APPLY : $(F, h) \mapsto F(h)$ is $([\delta_C \to \rho], \delta_C, \rho)$-computable,
- **(S)** if for some representation δ of a subset of $C(C[0;1], \mathbb{R})$, APPLY is (δ, δ_C, ρ)-computable then $\delta \leq [\delta_C \to \rho]$.

(**U**) corresponds to the "universal Turing machine theorem" and (**S**) to the "smn-theorem" from computability theory. Roughly speaking, $[\delta_C \to \rho]$ is the "poorest" representation of the set $C(C[0;1], \mathbb{R})$ for which the APPLY function becomes computable.

For converting the classical proof mentioned in Section 2 we needed a representation of the set $B[0;1]$ of bounded functions $g : [0;1] \to \mathbb{R}$. Since it has a cardinality bigger than that of Σ^ω, it has no representation. To overcome this difficulty it would suffice to extend F to the Banach space $B_1[0;1]$ generated by the continuous functions and all the characteristic function $\chi_{[0;x]}$, $0 \leq x \leq 1$. However, since this space is not separable we do not know any reasonable representation of it. We solve the problem by (implicitly) extending F only to functions $\chi_{[0;x]}$ from a countable dense set of points x in which g is continuous and for which we can compute $g(x) := \overline{F}(\chi_{[0;x]})$ from F and $\|F\|$. Remember that every function of bounded variation has at most countably many points of discontinuity.

Finally, for formulating a computable version of the Riesz representation theorem we need a representation for functions of bounded variation. In our context the only application of a function g of bounded variation is to compute the Riemann-Stieltjes integral $\int h\,\mathrm{d}g$ for continuous functions h. By Corollary 6, it suffices to know g on a countable dense set containing 0 and 1. Therefore it will suffice to consider only functions from BV with countable domain.

Definition 17. Let $\mathrm{BVC} := \{g \in \mathrm{BV} \mid \mathrm{dom}(g) \text{ is countable}\}$. Define a representation $\delta_{\mathrm{BVC}} :\subseteq \Sigma^\omega \to \mathrm{BVC}$ as follows: $\delta_{\mathrm{BVC}}(p) = g$ iff there are sequences $p_0, q_0, p_1, q_1, \ldots \in \Sigma^\omega$ such that $p = \langle\langle p_0, q_0\rangle, \langle p_1, q_1\rangle, \ldots\rangle$, $\rho(p_0) = 0$, $\rho(p_1) = 1$ and $\mathrm{graph}(g) = \{(\rho(p_i), \rho(q_i)) \mid i \in \mathbb{N}\}$.

Informally, a δ_{BVC}-name of g is a list of its graph. For proving computability of multi-functions on represented sets we use "generalized Turing machines" (GTMs) [Tavana and Weihrauch(2011)]. We call a generalized Turing machine M on represented sets *computable*, if all multi-functions on the represented sets occurring in M are computable. We use the following result: the multi-function f_M computed by a computable GTM M on represented sets is computable.

For a representation $\delta :\subseteq \Sigma^\omega \to Z$ a subset $Y \subseteq Z$ is δ-r.e., iff there is a Type-2 machine N such that for all $p \in \mathrm{dom}(\delta)$,

$$N \text{ halts on input } p \iff \delta(p) \in Y\,.$$

And $Y \subseteq Z$ is δ-decidable, iff Y and $Z \setminus Y$ are δ-r.e. [Weihrauch(2000)]. As an example, $x < y$ for real numbers is $[\rho, \rho]$-r.e.

5 The computable Riesz representation theorem

In the following "computable", "recursively enumerable" and "decidable" means computable, recursively enumerable and decidable, respectively, w.r.t. the notations and multi-representations mentioned in Section 4.

First, from F and $\|F\|$ we will compute some $g \in \mathrm{BVC}$ such that $F(h) = \int h\mathrm{d}g$. By the next lemma for every rational interval I we can compute subintervals J with arbitrarily small $\|F\|_J$.

Lemma 18. *There is a computable multi-function*

$$e : (F, z, I, n) \rightrightarrows J$$

that maps every continuous linear functional $F : C[0;1] \to \mathbb{R}$, its norm z, every open rational interval $I = (a;b) \subseteq [0;1]$ and every $n \in \mathbb{N}$ to some open rational interval J such that $\overline{J} \subseteq I$, $\mathrm{length}(J) \leq 2^{-n}$ and $\|F\|_J \leq 2^{-n}$.

Precisely speaking, the multi-function e is $([\delta_C \to \rho], \rho, \nu_I, \nu_{\mathbb{N}}, \nu_I)$ - computable.

Proof: By Lemma 9 there is some $x \in I$ such that $x \in \mathrm{PC}_F$. By Definition 7 there is some J, $x \in J \in \mathrm{RI}$, such that $\overline{J} \subseteq I$, $\mathrm{length}(J) \leq 2^{-n}$ and $\|F\|_J \leq 2^{-n}$. We show that the multi-function e is $([\delta_C \to \rho], \rho, \nu_I, \nu_{\mathbb{N}}, \nu_I)$-computable.

For F, $z = \|F\|$, $I = (a; b)$, $n \in \mathbb{N}$, $J \in \mathrm{RI}$ and $\overline{f} \in \mathrm{Pg}$ consider the conditions

$$\overline{J} \subseteq I, \quad \mathrm{length}(J) \leq 2^{-n}, \tag{20}$$

$$\overline{f}(x) = 0 \text{ for } x \in J\,, \tag{21}$$

$$\|\overline{f}\| \leq 1\,, \tag{22}$$

$$|F(\overline{f})| > \|F\| - 2^{-n}\,. \tag{23}$$

Conditions (20-22) are decidable (relative to their representations). Since $x < y$ is $[\rho, \rho]$-r.e. and $(F, \overline{f}) \mapsto F(\overline{f})$ is computable, (23) is r.e. Therefore, here is a Type 2-machine M that halts on input $(p_1, p_2, u_3, u_4, u_5, u_6)$ iff

$(F, \|F\|, I, n, J, \overline{f}) := ([\delta_C \to \rho], \rho, \nu_I, \nu_{\mathbb{N}}, \nu_I, \nu_I, \nu_{\mathrm{Pg}})(p_1, p_2, u_3, u_4, u_5, u_6)$ satisfies (20-23). From M a Type-2 machine N can be constructed which on input (p_1, p_2, u_3, u_4) (by the usual step counting technique) searches for (u_5, u_6) such that M halts on input $(p_1, p_2, u_3, u_4, u_5, u_6)$.

First we show that $J = \nu_I(u_5)$ and $\overline{f} = \nu_{\mathrm{Pg}}(u_6)$ exist.

Since Pg is dense in $C[0;1]$, $\|F\| = \sup\{|F(h)| \mid h \in \mathrm{Pg},\ \|h\| \leq 1\}$. Therefore, there is a function $h \in \mathrm{Pg}$ with $\|h\| \leq 1$ such that $|F(h)| > \|F\| - 2^{-n-1}$. As we have shown (replace above n by $n+1$) there is a rational interval $L \subseteq I$ such that $\mathrm{length}(L) \leq 2^{-n}$ and $\|F\|_L \leq 2^{-n-1}$. Let $(a_2; b_2) \subseteq L$ such that h has no vertex in $(a_2; b_2)$. Let $a_1 := a_2 + (b_2 - a_2)/3$, $b_1 := b_2 - (b_2 - a_2)/3$) and $J := (a_1; b_1)$. Define a function $f_0 \in \mathrm{Pg}$ by its vertices as follows:

$$(0,0), (a_2, 0), (a_1, h(a_1)), (b_1, h(b_1)), (b_2, 0), (1, 0)$$

and let $\overline{f} := h - f_0$. Then $\|f_0\| \leq 1$ and $|F(f_0)| \leq 2^{-n-1}$ since $\mathrm{NZ}(f_0) \subseteq L$. Since h and f_0 have no vertex in the interval $(a_2; a_1)$, $|h(x) - f_0(x)| \leq |h(a_2)| \leq 1$ for $a_2 \leq x \leq a_1$, correspondingly $|h(x) - f_0(x)| \leq 1$ for $b_1 \leq x \leq b_2$, and $|h(x) - f_0(x)| = 0$ for $a_1 \leq x \leq b_1$. We obtain $\|\overline{f}\| \leq 1$. Furthermore,

$$|F(\overline{f})| = |F(h - f_0)| \geq |F(h)| - |F(f_0)| \geq \|F\| - 2^{-n}\,.$$

Therefore, J and $\overline{f}$ exist.

It remains to show that J has the properties requested in the lemma. Obviously, $\overline{J} \subseteq I$ and $\mathrm{length}(J) \leq 2^{-n}$. Suppose $h \in C[0;1]$, $\|h\| \leq 1$ and $\mathrm{NZ}(h) \subseteq J$. Since $\mathrm{NZ}(h)$ and $\mathrm{NZ}(\overline{f})$ are disjoint and of norm ≤ 1, by Lemma 8, $|F(h)| + |F(\overline{f})| \leq \|F\|$ hence $|F(h)| \leq \|F\| - |F(\overline{f})| < 2^{-n}$. Therefore, $\|F\|_J \leq 2^{-n}$. $\square$

By iterating the function e from Lemma 18 in every open rational interval we can find some point $x \in \mathrm{PC}_F$ and the value $g_F(x)$.

Lemma 19. *The multi-function* $G : (F, \|F\|, I) \rightrightarrows (x, g_F(x))$ *mapping* F*, its norm and an interval* $I \in \mathrm{RI}$ *to* $(x, g_F(x))$ *for some* $x \in I \cap \mathrm{PC}_F$ *is computable.*

Proof: Let $J_{-1} := I$. For every $n \in \mathbb{N}$ let J_n be a result of applying the multi-function e from Lemma 18 to $(F, \|F\|, J_{n-1}, n)$. Then $(J_n)_{n\in\mathbb{N}}$ is a properly nested sequence of intervals with $\mathrm{length}(J_n) \leq 2^{-n}$. It converges to some point $x \in I$. Since for all n, $x \in J_n$ and $\|F\|_{J_n} \leq 2^{-n}$, $x \in \mathrm{PC}_F$. Furthermore, by Lemma 13, $|g_F(x) - F(s_{J_n})| \leq 2^{-n}$. Therefore $(F(s_{J_n}))_{n\in\mathbb{N}}$ converges fast to $g_F(x)$.

Let M_1 be a computable GTM computing the multi-function e from Lemma 18. From M_1 we can construct a computable GTM that on input $(F, \|F\|, I, n)$ computes in turn some $J_0, J_1, \ldots, J_n$ and then $(J_n, F(s_{J_n}))$ as its result.

By [Weihrauch(2008), Theorem 35] the multi-function $(F, \|F\|, I) \rightrightarrows (J_n, F(s_{J_n}))_{n\in\mathbb{N}}$ is computable (where the canonical representation considered for sequences [Weihrauch(2000)]). Since the limit operations for nested sequences of intervals converging to a point and for fast converging Cauchy sequences of real numbers are computable [Weihrauch(2000)], $(x, g_F(x))$ can be computed from $(J_n, F(s_{J_n}))_{n\in\mathbb{N}}$. Therefore, the multi-function G is computable. □

We can now prove our computable version of the Riesz representation theorem.

Theorem 20 (computable Riesz representation).
The multi-function $\mathrm{RRT} : (F, \|F\|) \rightrightarrows g$ *mapping every functional* $F : C[0;1] \to \mathbb{R}$ *and its norm to some function* $g \in \mathrm{BVC}$ *such that*
— $F(h) = \int h\mathrm{d}g$ *(for all* $h \in C[0;1]$*),*
— g *is continuous on* $\mathrm{dom}(g) \setminus \{0, 1\}$*,*
— $g(0) = 0$ *and* $\|F\| = \mathrm{Var}(g)$
is $([\delta_C \to \rho], \rho, \delta_{\mathrm{BVC}})$*-computable.*

Proof: Let $L_0, L_1, \ldots$ be a canonical numbering of the set RI of open rational intervals. By Lemma 19 there is a computable function G' mapping $(F, \|F\|, n)$ to some $(x_n, y_n) \in \mathbb{R}^2$ where $(x_0, y_0) = (0, 0)$, $(x_1, y_1) = (1, F(1))$ and $(x_n, y_n)) \in G(f, \|F\|, L_n)$ if $n \geq 2$. Since $x_n \in \mathrm{PC}_F$ and $y_n = g_F(x_n)$ for all $n \geq 2$, $\{(x_n, y_n) \mid n \in \mathbb{N}\}$ is the graph of a restriction g of g_F. Since $\{x_n \mid n \in \mathbb{N}\}$ is dense, $g \in \mathrm{BVC}$. By Theorem 15, g is continuous on $\mathrm{dom}(g) \setminus \{0, 1\}$ and $\mathrm{Var}(g) = \|F\|$. Obviously, $g(0) = 0$. By Theorem 16, $F(h) = \int h\mathrm{d}g$ (for all $h \in C[0;1]$).

By the type conversion theorem [Weihrauch(2008), Theorem 33], the multi-function $(F, \|F\|) \rightrightarrows ((x_n, y_n))_{n\in\mathbb{N}}$ is $([\delta_C \to \rho], \rho, [\nu_\mathbb{N} \to [\rho, \rho]])$-computable. From a $[\nu_\mathbb{N} \to [\rho, \rho]]$-name of the sequence $((x_n, y_n))_{n\in\mathbb{N}} = ((x_0, y_0), (x_1, y_1), \ldots)$

we can compute a $[\rho,\rho]^\omega$- name [Weihrauch(2000), Lemma 3.3.16] which is a δ_{BVC}-name of g. $\square$

Finally, we prove that a reverse of the Riesz representation theorem is computable.

Theorem 21. *The operator $T:(g,l)\mapsto F$, mapping every $g\in\mathrm{BVC}$ and every $l\in\mathbb{N}$ with $Var(g)\le 2^l$ to the functional F defined by $F(h)=\int h\,\mathrm{d}g$ for all $h\in C[0;1]$, is computable.*

Proof: First we show that $(G,l,h)\mapsto\int h\,\mathrm{d}g$ is computable. By Theorem 6.2.7 in [Weihrauch(2000)] a modulus $m:\mathbb{N}\to\mathbb{N}$ of continuity of h can be computed from h. let ν_{fs} be a canonical notation of the finite sequences of natural numbers. The set of all $(g,(i_1,\dots,i_{n-1}),j)$ such that $(0,x_{i_1},\dots,x_{i_{n-1}},1)$ is a partition for g of precision j is $(\delta_{\mathrm{BVC}},\nu_{\mathrm{fs}},\nu_{\mathbb{N}})$-r.e. There is computable GTM on represented sets which on input (g,j) finds a sequence $(i_1,\dots,i_{n-1})$ such that $(0,x_{i_1},\dots,x_{i_{n-1}},1)$ is a partition for g of precision j. Therefore from (g,h,k,l) we can compute a sequence $(i_1,\dots,i_{n-1})$ such that $X:=(0,x_{i_1},\dots,x_{i_{n-1}},1)$ is a partition for g of precision $m(k+l+1)$. By Lemma 4, $|\int h\,\mathrm{d}g-S(g,h,X)|\le 2^{-l-k}V(g)\le 2^{-k}$. The function $(g,h,X)\mapsto S(g,h,X)$ is computable (by a computable GTM). Therefore, from (g,l,h,k) a number y_k can be computed (multi-valued) such that $|\int h\,\mathrm{d}g-y_k|\le 2^{-k}$. By [Weihrauch(2008), Theorem 33] the multi-function $(g,l,h)\rightrightarrows(y_k)_{k\in\mathbb{N}}$ is computable. By [Weihrauch(2000), Theorem 4.3.7], $(g,l,h)\to\int h\,\mathrm{d}g$ is $(\delta_{\mathrm{BVC}},\nu_{\mathbb{N}},\delta_C,\rho)$-computable. By [Weihrauch(2000), Theorem 3.3.15], $(g,l)\mapsto F$ such that $F(h)=\int h\,\mathrm{d}g$ is $(\delta_{\mathrm{BVC}},\nu_{\mathbb{N}},[\delta_C\to\rho])$-computable. $\square$

By Theorem 20, from F and $\|F\|$ we can compute g such that $\mathrm{Var}(g)=\|F\|$, and by Theorem 21, from g and an upper bound of $\mathrm{Var}(g)$ we can compute F.

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Appendix

Proof of Lemma 4

Since there are partitions for g of arbitrary precision, I is unique if it exists. Next, we prove

$$|S(g,h,Z_1) - S(g,h,Z_2)| \leq 2^{-k}V(g)\,. \tag{24}$$

for any two partitions Z_1, Z_2 for g with precision $m(k+1)$.

Let $Z_1 = (x_0, x_1, \ldots, x_n)$ and let Z' be a refinement of Z_1. Z' can be written as

$$x_0 = y_0^1, y_1^1, \ldots, y_{j_1}^1 = x_1 = y_0^2, y_1^2, \ldots, y_{j_2}^2 = x_2 \ldots\ \ldots = y_0^n, y_1^n, \ldots, y_{j_n}^n = x_n$$

$(j_1, \ldots, j_n \geq 1)$. Then

$$\begin{aligned}
&|S(g,h,Z_1) - S(g,h,Z')| \\
&= \left| \sum_{i=1}^{n} h(x_i)\big(g(x_i) - g(x_{i-1})\big) - \sum_{i=1}^{n}\sum_{l=1}^{j_i} h(y_l^i)\big(g(y_l^i) - g(y_{l-1}^i)\big) \right| \\
&= \left| \sum_{i=1}^{n} h(x_i) \sum_{l=1}^{j_i} \big(g(y_l^i) - g(y_{l-1}^i)\big) - \sum_{i=1}^{n}\sum_{l=1}^{j_i} h(y_l^i)\big(g(y_l^i) - g(y_{l-1}^i)\big) \right| \\
&= \left| \sum_{i=1}^{n}\sum_{l=1}^{j_i} \big(h(x_i) - h(y_l^i)\big)\big(g(y_l^i) - g(y_{l-1}^i)\big) \right| \\
&\leq \sum_{i=1}^{n}\sum_{l=1}^{j_i} \big|h(x_i) - h(y_l^i)\big|\ \ \big|g(y_l^i) - g(y_{l-1}^i)\big| \\
&\leq 2^{-k-1} \sum_{i=1}^{n}\sum_{l=1}^{j_i} \big|g(y_l^i) - g(y_{l-1}^i)\big| \qquad \text{since } |x_i - y_l^i| \leq 2^{-m(k+1)} \\
&\leq 2^{-k-1}V(g)
\end{aligned}$$

Now let Z' be a common refinement of Z_1 and Z_2. Then $|S(g,h,Z_1) - S(g,h,Z_2)| \leq |S(g,h,Z_1) - S(g,h,Z')| + |S(g,h,Z') - S(g,h,Z_2)| \leq 2^{-k}V(g)$.

There is a sequence $(Z_k)_k$ of partitions for g such that Z_k has precision $m(k+1)$. By (24) for $j > k$, $|S(g, h, Z_k) - S(g, h, Z_j)| \leq 2^{-k}V(g)$. Let I be the limit of the Cauchy sequence $(S(g, h, Z_k))_k$. Let Z be a partition of precision $m(k+1)$. Then for every $i > k$ by (24),

$$\begin{aligned}|I - S(g, h, Z)| &\leq |I - S(g, h, Z_i)| + |S(g, h, Z_i) - S(g, h, Z)| \\ &\leq 2^{-i}V(g) + 2^{-k}V(g)\,,\end{aligned}$$

hence $|I - S(g, f, Z)| \leq 2^{-k}V(g)$. $\square$