Journal of Universal Computer Science, vol. 4, no. 2 (1998), 193-201
submitted: 25/9/97, accepted: 1/11/97, appeared: 28/2/98 © Springer Pub. Co.

Numerical Verification Method of Existence of Connecting Orbits for Continuous Dynamical Systems

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Abstract: In this paper, a numerical method is presented for proving the existence and inclusion of connecting orbits of continuous dynamical systems described by parameterized nonlinear ordinary differential equations. Taking a certain second order nonlinear ordinary differential equaiton as an example, the existence of homoclinic bifurcation points is proved by the method.

Key Words: Connecting Orbits, Defining Equation of Stable-Manifolds, Numerical Verification of Existence of Nonlinear Boundary Value Problems

1 Introduction

In this paper, we are concerned with the parameterized nonlinear dynamical system described by

$$\frac{dx}{dt} = f(x, \lambda),\ x(t) \in R^n,\ \lambda \in R^p. \tag{1}$$

Here, $f(x, \lambda)$ is a continuously differentiable map from R^{n+p} to R^n. An orbit $x(t)^*$, $-\infty < t < \infty$ of Eq.(1) with $\lambda = \lambda^*$ is called a connecting orbit if the limits

$$x_- = \lim_{t \to -\infty} x(t),\ x_+ = \lim_{t \to \infty} x(t) \tag{2}$$

exits, where x_- and x_+ are equilibrium points of Eq.(1), i.e., they are zero points of $f(x, \lambda^*) = 0$. A connecting orbit is called a homoclinic orbit and a heteroclinic orbit if $x_- = x_+$ and $x_- \neq x_+$, respectively.

Since the connecting orbits play important roles for the analysis of the global behavior of nonlinear dynamical systems, numerical methods for calculating them have been studied. Especially, since 1990 the methods based on the projection boundary condition have been proposed [1],[2]. The stability and the convergence property of such numerical methods have also been proved. However, a posteriori error analysis for such methods has not presented yet.

The aim of this paper is to present a numerical method for verification of the existence and inclusion of connecting orbits for Eq.(1). That is, in association with an approximate connecting orbit obtained using the projection boundary condition, we present an algorithm which may answer the question as to whether there exits an exact connecting orbit in some neighborhood of an approximate one, and in the affirmative case may give a bound between an approximate one and exact one.

2 Formulation as Boundary Value Problem

In this section, the problem of obtaining connecting orbits of Eq.(1) is reduced to its boundary value problem.

Let $x_- = x_-(\lambda)$ and $x_+ = x_+(\lambda)$ be hyperbolic equiliblium points of Eq.(1). Let $W^s(x_-)$ and $W^s(x_+)$ be stable manifolds of $x_- = x_-(\lambda)$ and $x_+ = x_+(\lambda)$, respectively and $W^u(x_-)$, $W^u(x_+)$ unstable manifolds of $x_- = x_-(\lambda)$ and $x_+ = x_+(\lambda)$, respectively. Let n^s_-, n^s_+ $n^u_- = n - n^s_-$, and $n^u_+ = n - n^s_+$ are dimensions of such manifolds, respectively. We assume that the dimensions of these manifolds are constants within the given range of λ and satisfy

$$n + 1 = n^u_- + n^s_+ + p. \tag{3}$$

Since for homoclinic orbits the relation $n^u_- + n^s_+ = n$ holds, we have $p = 1$ for this case.

Let us further define $-\infty < T_- < T_+ < \infty$ and $J = [T_-, T_+]$. If the local unstable manifold $W^u(x_-)$ and the local stable manifold $W^s(x_+)$ are defined by the following equations:

$$B_-(x(T_-), \lambda) = 0,\ B_- : R^n \times R^p \to R^{n-n^s_-}, \tag{4}$$

and

$$B_+(x(T_+), \lambda) = 0,\ B_+ : R^n \times R^p \to R^{n-n^u_+}, \tag{5}$$

respectively. Then the conecting orbit starting from x_- and ending with x_+ becomes a solution of the following boundary value proble:

$$\begin{cases} \frac{dx}{dt} = f(x, \lambda) \\ \frac{d\lambda}{dt} = 0, \\ B_-(x(T_-), \lambda) = 0, \\ B_+(x(T_+), \lambda) = 0, \\ \Psi(x, \lambda) = 0. \end{cases} \tag{6}$$

Here,$\Psi : C^1(J, V) \times R^p \to R$. The final equation of (1)is called a phase condition. For example, as a phase condition the following is often used:

$$\Psi(x, \lambda) = s^T[x(0) - v] = 0, \tag{7}$$

where $s, v \in R^n$ are constant vectors.

If we take Eq.(7) as a phase condition, then Eq.(6) becomes a three point boundary value problem. In what follows, we normalize the time variable such that the time interval is given by $J = [-1, 1]$.

3 Defining Equation of Local Stable and Unstable Manifold

In this section, we write down explicitly the defining equation of local unstable manifold $W^u(x_-)$ and local stable manifold $W^s(x_+)$:

$$B_-(x(T_-), \lambda) = 0,\ B_- : R^n \times R^p \to R^{n-n^s_-}, \tag{8}$$

and

$$B_+(x(T_+), \lambda) = 0,\ B_+ : R^n \times R^p \to R^{n-n^u_+}, \tag{9}$$

respectively. Here, we adopt Perron's integral equation method.

Let $x_c(\lambda)$ be a family of hyperbolic equilibrium points satisfying $f(x_c(\lambda),\lambda) = 0$. Assume that $x_c(\lambda)$ is a continuous function of λ. By making use of the coordinate transformation

$$u = x - x_c(\lambda), \tag{10}$$

we shift these equilibrium points to the origine. If we put

$$A(\lambda) = f_x(x_c(\lambda),\lambda), \tag{11}$$

then Eq.(1) is rewritten as

$$\frac{du}{dt} = A(\lambda)u + s(u,\lambda). \tag{12}$$

Here,

$$s(u,\lambda) = f(u + x_c(\lambda),\lambda) - f_x(x_c(\lambda),\lambda)u. \tag{13}$$

Assume that in the given range of λ, real parts of k eigenvalues of $A(\lambda)$ are negative and remaining $n-k$ are positive. Then we can choose a regular matrix $P(\lambda)$ such that

$$P(\lambda)A(\lambda)P(\lambda)^{-1} = B(\lambda) = \begin{pmatrix} B_1(\lambda) & 0 \\ 0 & B_2(\lambda) \end{pmatrix}. \tag{14}$$

Here, $B_1(\lambda)$ is $k \times k$ matrix whose all eigenvalues have negative real parts and $B_2(\lambda)$ is a $(n-k)\times(n-k)$ matrix whose all eigenvalues have positive real parts. Put

$$v = P(\lambda)u. \tag{15}$$

Then Eq.(12) becomes

$$\frac{dv}{dt} = B(\lambda)v + S(v,\lambda). \tag{16}$$

Here,

$$S(v,\lambda) = P(\lambda)s(P(\lambda)^{-1}v,\lambda). \tag{17}$$

In what follows, for a given range of λ we assume that for any positive ϵ we can take sufficiently small δ such that if $|v_1|,|v_2| \le \delta$ implies

$$|S(v_1,\lambda) - S(v_2,\lambda)| \le \epsilon|v_1 - v_2|. \tag{18}$$

If we put

$$U_1(t,\lambda) = \begin{pmatrix} e^{tB_1(\lambda)} & 0 \\ 0 & 0 \end{pmatrix},\ U_2(t,\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & e^{tB_2(\lambda)} \end{pmatrix}, \tag{19}$$

we have

$$e^{tB(\lambda)} = U_1(t,\lambda) + U_2(t,\lambda) \tag{20}$$

and

$$\frac{dU_i(\lambda)}{dt} = B(\lambda)U_i(\lambda),\ i = 1,2. \tag{21}$$

We assume that for a geven range of λ, all eigenvalues of $B_1(\lambda)$ have real parts less than $-\alpha$, $\alpha > 0$. Then, we can take sufficiently small positive number σ such that real parts of eigenvalues of $B_1(\lambda)$ are less than $-(\alpha+\sigma)$ and those of eigenvalues of $B_2(\lambda)$ are greater than σ. Thus we can take suffficinetly large number K such that for $t \geq 0$

$$|U_1(t, \lambda))| \leq Ke^{-(\alpha+\sigma)t} \tag{22}$$

holds and for $t \leq 0$

$$|U_2(t, \lambda)| \leq Ke^{\sigma t} \tag{23}$$

hold.

Put

$$v(T) = v_T = (v_{T_1}, v_{T_2}, \cdots, v_{T_n})^T. \tag{24}$$

We define

$$v_T^1 = (v_{T_1}, v_{T_2}, \cdots, v_{T_k}, 0, 0, \cdots, 0)^T \in R^n, \tag{25}$$

and

$$v_T^2 = (0, 0, \cdots, 0, v_{Tk+1}, v_{Tk+2}, \cdots, v_{Tn})^T \in R^n. \tag{26}$$

Then we introduce the following integral equation:

$$\begin{aligned}\theta(t, \lambda, v_T^1) = U_1(t-T, \lambda)v_T^1 &+ \int_{T_+}^{t} U_1(t-s, \lambda)S(\theta(s, \lambda, v_T^1))ds \\ &- \int_t^\infty U_2(t-s, \lambda)S(\theta(s, \lambda, v_T^1))ds. \end{aligned} \tag{27}$$

Using the contraction mapping theorem in the suitable Banach space it is seen that if $|v_T^1|$ is sufficiently small, the solution of this integral equation exists uniquely. Using this solution, we can define

$$\psi_+(v_T^1, \lambda) = -\int_{T_+}^\infty U_2(T-s, \lambda)S(\theta(s, \lambda, v_T^1))ds, \tag{28}$$

since it is seen also that if $|v_T^1|$ is sufficiently small, this integral exits. The function ψ_+ becomes as smooth as S. The defining equation of local stable manifold of $x_c(\lambda)$ is given by

$$B_+(x(T_+), \lambda) = v_T^2 - \psi_+(v_T^1, \lambda) = 0. \tag{29}$$

The defining equation of the local unstable manifold of $x_c(\lambda)$ can be derived using the coordinate change $t' = -t$, since this change reduces the defining equation of the local unstable manifold to that of stable one.

4 Numerical Existence Theorem

Let $Y = C[-1,1]$ be the Banach space of continuous $n+p$-dimensional vector valued functions $y(t) = (x_1(t), x_2(t), \cdots, x_n(t), \lambda_1(t), \lambda_2(t), \cdots, \lambda_m(t))$ on the interval $J = [-1,1]$. We use the scaled maximum norm

$$\|y\|_u = \sup_{t \in I} |y(t)|_u. \tag{30}$$

Here,

$$|y(t)|_u = \max_{1 \le i \le m} \frac{|y_i(t)|}{u_i}, \tag{31}$$

where $m = n + p$ and $u = (u_1, u_2, \cdots, u_m)$ is a constant m-dimensional vector with positive elements, $u_i > 0$ for $i = 1, 2, \cdots, m$. Let $D = C^1[-1,1]$ be the Banach space of m-dimensional vector valued functions $y(t) = (y_1(t), y_2(t), \cdots, y_m(t))$ such that

$$y \in C[-1,1] \text{ and } \frac{dy}{dt} \in C[-1,1]. \tag{32}$$

Moreover, let $C^1[-1,1;M]$ be the Banach space of $m \times m$ matrix valued C^1 function on J.

In the following we assume that the given approximate connecting orbit pair $c = (x_a, \lambda_a)$ is an element of Y. Now we define $F : D \to Y$ by

$$Fy = \left(\frac{dy}{dt} - e(y), g(y) \right), \tag{33}$$

where

$$e(y) = \begin{pmatrix} f(x,\lambda) \\ 0 \end{pmatrix} \tag{34}$$

and

$$g(y) = \begin{cases} B_-(x(T_-), \lambda) \\ B_+(x(T_+), \lambda) \\ \Psi(x, \lambda) \end{cases}. \tag{35}$$

Then the original boundary value problem can be written as

$$Fy = 0. \tag{36}$$

In what follwos, we assume that $e : D \to Y$ and $g : D \to R^m$ are continuously Fréchet differentiable. Then $F : D \to Y \times R^m$ becomes Fréchet differentiable and its Fréchet derivative $D_y F(y) : D \to Y \times R^m$ is given by

$$D_y F(y) h = \left(\frac{dh}{dt} - e_y(y) h, D_y g(y) h \right), \tag{37}$$

where $h \in D$. Let an $m \times m$ matrix function $\tilde{A}(t)$ and a vector valued linear functional l be approximations of $e_y(c)$ and $D_y g(c)$, respectively. We define

$$\tilde{L} h = \left(\frac{dh}{dt} - \tilde{A}(t) h, lh \right). \tag{38}$$

Let $\tilde{\Phi}(t)$ be a fundamental matrix of

$$\frac{dz}{dt} = \tilde{A}(t)z \tag{39}$$

satisfying

$$\tilde{\Phi}(-1) = I. \tag{40}$$

Since this equation is time varing, usually this fundamental matrix can be obtained approximately using the numerical computation. Let $\Phi(t) \in C^1[-1,1;M]$ be such an approximate fundamental matrix satisfying

$$\Phi(-1) = I. \tag{41}$$

Assuming that $\Phi(t)$ is invertible for all $t \in J$, we put

$$A(t) = \frac{d\Phi(t)}{dt}\Phi^{-1}(t). \tag{42}$$

Then, we have

$$\frac{d\Phi(t)}{dt} = A(t)\Phi(t) \tag{43}$$

and

$$\Phi(-1) = I. \tag{44}$$

Thus we can define

$$Lh = \left(\frac{dh}{dt} - A(t)h, lh\right) \text{ for} h \in D. \tag{45}$$

The following lemma is due to Urabe[3]:
Lemma(Urabe) Let $\Phi(t)$ be a fundamental matix of

$$\frac{dz}{dt} = A(t)z, \tag{46}$$

satisfying

$$\Phi(-1) = I. \tag{47}$$

Let $G = l[\Phi(t)]$ be a matrix whose column vector is given by $l[\phi_i(t)]$, $i - 1, 2, \cdots, m$. Here, $\phi_i(t)$ is a vector defined as the i-th column vector of the matrix $\Phi(t)$. Then G is invertible iff the linear operator L defined by (38)is invertible. In case of G is invertible, we have

$$L^{-1}(\phi, u) = H\phi + Su, \tag{48}$$

where $\phi \in X$, $u \in R^n$, and H is a linear operator from X to $D \subset X$ defined by

$$H\phi = \Phi(t)\int_{-1}^{t}\Phi^{-1}(s)\phi(s)ds - \Phi(t)G^{-1}l[\Phi(t)\int_{-1}^{t}\Phi^{-1}(s)\phi(s)ds]. \tag{49}$$

Here S is a linear operator from R^n to D given by $Sv = \Phi(t)G^{-1}v$. □

Assuming G is invertible, we define a Newton-like operator $k : X \to X$ by

$$k(y) = L^{-1}(L - F)x = L^{-1}(e(y) - A(t)y, l(y) - g(y)). \tag{50}$$

It is seen that if $y^* \in Y$ is a fixed point of k, it becomes an element of D and satisfies $Fy^* = 0$.

In order to verify numerically that the operator k has a fixed point on a ball around the given approximate solution, we introduce an infinite-dimenstional version of Krawczyk operator. For this purpose, we use the theory of interval functions. An interval function $Y(t)$on J is defined by

$$Y(t) = [\underline{y}(t), \overline{y}(t)]. \tag{51}$$

The functions $\underline{y}(t)$ and $\overline{y}(t)$ are called end point functions. In this paper, we assume that endpoint functions are elements of $C[-1,1]$ and interval function $Y(t)$ is a set of functions $y \in C[-1,1]$ satisfying $\underline{y}(t) \le y(t) \le \overline{y}(t)$.

The addition, subtraction, multiplication and division between two interval functions are defined pointwisely. The integration of an interval function $Y(t)$ is defined by

$$\int_{-1}^{t} Y(s)ds = \left[\int_{-1}^{t} \underline{y}(s)ds, \int_{-1}^{t} \overline{y}(s)ds\right]. \tag{52}$$

Here the integrations in the right hand side are the Riemann integals.

A vector or a matrix valued interval function is defined as a vector or a matrix whose elements are interval functions, respectively. If $Y(t)$ is a vector or a matrix valued interval function, $|Y(t)|$ is defined as a vector or a matrix valued interval function whose elements are given by $|Y_i(t)|$ or $|Y_{ij}(t)|$, respectively. Here, for an interval $[a, b]$

$$|[a, b]| = \max(|a|, |b|). \tag{53}$$

The Mid-function is defined by

$$\mathrm{Mid}(Y(t)) = \frac{\overline{y}(t) + \underline{y}(t)}{2}. \tag{54}$$

Let $T(t)$ be an interval function such that Mid $(T(t)) = c(t)$. We define the following Krawczyk operator:

$$K(T) = k(c) + M(T - c), \tag{55}$$

where

$$M = L^{-1}(L - DF(T)) \text{ and } c = \mathrm{Mid}(T). \tag{56}$$

Then we have the following theorem:

Theorem Let $T(t)$ be an interval function such that Mid $(T(t)) = c(t)$. If

$$K(T(t)) \subset T(t) \tag{57}$$

and

$$\|M\|_u < 1, \tag{58}$$

hold, then there exist a fixed point y^* of k uniquely in $T(t) \subset y$. This y^* is an element of D and satisfies $Fy^* = 0$. Moreover, $DF(y^*)$ is invertible. □

Remark The condition (58) is satisfied if $K(T(t))$ is a proper subset of $T(t)$.

The condition (57) is usually verified in the following form:

$$K(T(t)) - c(t) \subset T(t) - c(t). \tag{59}$$

□

Here, we note the following property:
[**Property**] Let A be an $m \times n$ interval matrix and B an $n \times p$ interval matrix. If Mid $B = 0$,

$$AB = |A|B = [-|A||B|, |A||B|] = [-1, 1]|A||B|. \tag{60}$$

□

¿From this property, noticing Mid $(T - c) = 0$, we have

$$\begin{aligned} M(T(t) - c) &= \Phi(t) \int_{-1}^{t} \Phi^{-1}(s)R(s)(T(s) - c(s))ds \\ &- \Phi(t)G^{-1}l[\Phi(t) \int_{-1}^{t} \Phi^{-1}(s)R(s)(T(s) - c(s))ds] \\ &+ \Phi(t)G^{-1}(l - Dg(T(t))(T(t) - c(t)) \\ &= [-1, 1]|\Phi(t)| \int_{-1}^{t} |\Phi^{-1}(s)||R(s)||T(s) - c(s)|ds \\ &- \Phi(t)G^{-1}l[[-1, 1]|\Phi(t)| \int_{-1}^{t} |\Phi^{-1}(s)||R(s)||T(s) - c(s)|ds] \\ &+ \Phi(t)G^{-1}(l - Dg(T))(T(t) - c(t)), \end{aligned} \tag{61}$$

where

$$R(t) = De(T) - A(t). \tag{62}$$

5 Example

In this section, as an example, we consider the following nonlinear ordinary differential equation:

$$\frac{dx}{dt} = y, \ \frac{dy}{dt} = x - x^2 + \lambda y + 0.5xy. \tag{63}$$

Equilium points are $(0, 0)$ and $(1, 0)$. This equation has a super-critical Hopf bifurcation point at $\lambda = -0.5$ on $(1, 0)$.

If the parameter λ is increased from this value, period of the generated periodic solution bocomes longer and longer and finally the periodic solution becomes a homoclinic orbit of the saddle $(0, 0)$ at certain value of λ. If further increse the value of λ then there is no periodic solution nor homoclinic orbit. This is a homoclinic bifurcation phenomena. We have tried to verify the existence of a homoclinic bifurcation point λ_c and a homoclinic orbit.

We set $T_+ = -T_- = 10$. An approximate homoclinic orbit pair is seeked under the projection boundary value conditions[1],[2] using Urabe's Chebyshev polynomial expansion method[4]. An approximate solution is obtained for x as 90-th order Chebyshev polynomial. As a result, as an approximation of λ_c, we have

$$\lambda_c = -\frac{969926592}{2258238379} = -0.429505848904\cdots. \tag{64}$$

Moreover, as an approximation of the homoclinic orbit, we have an orbit shown in Fig.1.

Using this approximate homoclinic orbit pair, we have tried to verify the existence of the true homoclinic orbit pair numerically based on the method presented in the previous sections. Then, we have succeeded in proving the existence of the true homocilinic orbit pair and found that the error of the value of an approximate homoclinic point is at most 0.0000000022. Therefore, we have an inclusion of the homoclinic bifurcation point as

$$\lambda_c \in [-0.429505852, -0.429505846]. \tag{65}$$

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