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An Introduction To Polypodic Structures

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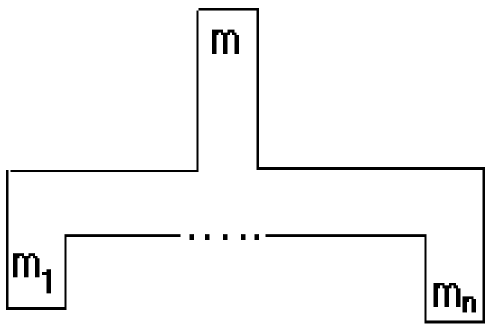
Abstract: Pollypodes is an algebraic structure in between monoids and $\Gamma - a\lg ebras$ having the advantages of both of them.Many objects of different nature such as words, trees, graphs, functions, etc, can be studied in common into the framework of polypodes.

1 Introduction

Substitution is a basic operation in Computer Science consisting of producing new objects by inserting objects of a certain kind into another object. Formally, it can be described by a function of the type

$M \times M^n \to M \quad , \quad (m, m_1, ..., m_n) \mapsto m[m_1, ..., m_n]$

called **polypodic operation** because it combines elements with $n - tuples$ of elements to get elements:



Polypodes are sets endowed with such an operation fulfilling asscociativity and unit axiom. There are at least four reasons motivating the introduction and study of these new structures.

1)Polypodes provide an ideal set up to study in common objects of different nature such as words , trees, graphs, functions, etc.Polypodes of functions were introduced in P.Cohn under the name of clones (cf. [Cohn 65]).

2)Polypodes seems to be the right structure needed to establish Eilemberg's famous theorem on varieties into the framework of trees.(cf [Bozapalidis2 99]).The reason is that in polypodes the operation domain is merged with the carrier set which is not the case of $\Gamma - a\lg ebras$ used by other authors as dominating algebraic structure.

3) Polypodes are close related to recursive program schemes.Actually , additive polypodes are used to solve systems of equations (where right hand side members are polynomials with coefficients in a semiring) into the space of $\omega - continuous$ functions over additive semimodules (cf [Bozapalidis1 99])

4)Polypodic calculus is a many variable calculus and so recognizability and rationality phenomena appear in a wide diversity namely global recognizability (which in monoids and trees coincides with ordinary recognizability) , local recognizability (which corresponds to rationality in monoids and equationality in $\Gamma - a\lg ebras$), partial recognizability (which is referred to substitution over specified variables), top recognizability (saturation under equivalences respecting the top argument) , etc.

The present paper is a short version of a part of [Bozapalidis1 99].It is composed of five sections.

The category of $n - polypodes$ is introduced in section 1.The set $T_\Gamma(X_n)$ of trees over the $n-ranked$ alphabet Γ and the set of variables $X_n = \{x_1, ..., x_n\}$ is the free $n-polypode$ generated by Γ.Three distinct notions of congruence appear according the arguments we use to define them :polypodic (resp. bottom,top) congruencies are equivalent relations compatible with the polypodic operation at all(resp. bottom ,top) arguments.Characterizations via monoid actions of the above kinds of congruencies are given.

In section 2 ,one exhibits remarkable examples of polypodes : words, trees, graphs, functions etc.Syntax in polypodes is examined in section 3.To each subset L of a polypode M its syntactical polypodic congruence $\equiv_L$ and polypode M_L are associated . The finite character of the above invariants leads to a notion of recognizability called **global .** Characterization through derivatives is also obtained. Section 4 is devoted to **locally** recognizable subsets which are projections of recognizable forests on polypodes . These subsets have all the nice properties of the rational subsets in monoids and equational subsets in $\Gamma - a\lg ebras$ .Finally in the last section we discuss top partial and projective recognizability

2 Polypodes

Polypodes are algebraic structures in between monoids and $\Gamma - a\lg ebras$ and they have the advantages of both of them.

Precisely , an $n - polypode$ (n$\geq$ 1) is a set M equipped with an operation

$$M \times M^n \to M \qquad (m, m_1, ..., m_n) \mapsto m[m_1, ..., m_n]$$

which is associative , i.e. for all $m, m_i, , m_j' \in M$ it holds

$m[m_1, ..., m_n][m_1', ..., m_n'] = m[m_1[m_1', ..., m_n'], ..., m_n[m_1', ..., m_n']]$

and has a unit , that is an n-tuple $e = (e_1, ..., e_n)$ fulfilling the axiom

$m[e_1, ..., e_n] = m$, $e_i[m_1, ..., m_n] = m_i$ for all $m, m_i \in M$.

A unit whenever exists it is unique. Monoids are $1-polypodes$ and each $n-polypode$ M can be viewed as a $\Gamma - a\lg ebra$ with $\Gamma = M$

Polypode morphism and subpolypodes are defined classically .Just a word for the subpolypode $pol(L)$ generated by a subset $L \subseteq M$; it is defined by the next items :

$-\{e_1, ..., e_n\} \cup L \subseteq pol(L)$

$-$if $m \in L$ and $m_1, ..., m_n \in pol(L)$, then $m[m_1, ..., m_n] \in pol(L)$

Free Object Consider an $n - ranked$ alphabet Γ and a set of variables $X_n = \{x_1, ..., x_n\}$. $T_\Gamma(X_n)$ as usual denotes the set of all trees constructed by Γ and indexed by X_n.For $t, t_1, ..., t_n \in T_\Gamma(X_n)$

, $t[t_1/x_1, ..., t_n/x_n]$ or shortly $t[t_1, ..., t_n]$ is the tree obtained by substituting t_i at all occurrences of x_i inside t $(1 \leq i \leq n)$

Theorem 1. *$T_\Gamma(X_n)$ with the above operation is the free $n-polypode$ generated by Γ.*

Remark. We should notice that the same set $T_\Gamma(X_n)$ is the free $\Gamma - a\lg ebra$ generated by X_n .

In polypodes there are three distinct notions of congruence.

An equivalence $\sim$ on a polypode M is said to be a **polypodic congruence** if

$m \sim m'$, $m_i \sim m_i'$, $i = 1, ..., n \quad \Rightarrow m[m_1, ..., m_n] \sim m'[m_1', ..., m_n']$

i.e. we have compatibility at all the arguments.

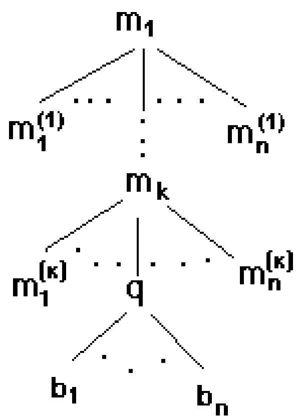
Bottom and Top congruence come by requiring compatibility at the bottom and top arguments respectively. Thus $\sim$ is a **bottom (resp.top) congruence** if

$m_i \sim m_i'$, $i = 1, ..., n \quad \Rightarrow m[m_1, ..., m_n] \sim m[m_i', ..., m_n']$ $\forall m$

$(resp.\ m \sim m' \Rightarrow m[m_1, ..., m_n] \sim m'[m_1, ..., m_n]$ $\forall m_i)$

Ordinary congruence on trees coincide with our bottom congruences.Thus given that a recognizable forest is a set of trees saturated by a finite index ordinary congruence ,it is quite natural to ask about the recognition power of polypodic congruencies.TheThe answer will be given later.

The above type of congruence can be characterized via monoid actioms.Let M be an $n-polypode$. We use the notation $\Delta_M^{(n)}$ for the set of all trees of the form



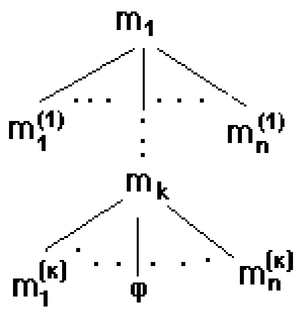
where ϕ is a symbol of rank n and $m_i, m_i^{(j)}, b_i$ belong to M.

Substitution at ϕ is indeed a multiplication in $\Delta_M^{(n)}$ converting this set into a monoid which canonically acts on M :

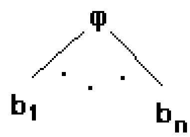
$\Delta_M^{(n)} \times M \to M \qquad (\tau, m) \mapsto \tau m$

Two submonoids of $\Delta_M^{(n)}$ are of interest:

$P_M^{(n)}$ and $R_M^{(n)}$. The elements of the first one are all trees of the form



and the elements of the second are all trees of the form



Theorem 2. *$\Delta_M^{(n)}$ is the direct product of $P_M^{(n)}$ and $R_M^{(n)}$. Further an equivalence relation $\sim$ on M is a polypodic (resp.bottom,top) congruence if and only if it is compatible with $\Delta_M^{(n)}$ (resp. $P_M^{(n)}$, $R_M^{(n)}$) action:*
$m \sim m' \Rightarrow \tau m \sim \tau m' \ \forall \tau \in \Delta_M^{(n)}$ *(resp.* $\tau \in P_M^{(n)}$, $R_M^{(n)}$)

The above result allows us to construct the least polypodic (resp. bottom , top) congruence including a relation. Precisely

Theorem 3. *Let M be a polypode and $R \subseteq M \times M$ .Then the polypodic congruence generated by R is given by*
$\langle R \rangle_{pol} = \{(\tau a, \tau b)/(a,b) \in \bar{R} \ , \ \tau \in \Delta_M^{(n)}\}$
Accordingly, the bottom (resp. top) congruence generated by R is given by
$\langle R \rangle_{bottom} = \{(\tau a, \tau b)/(a,b) \in \bar{R} \ , \tau \in P_M^{(n)}\}$
(resp. $\langle R \rangle_{top} = \{(\tau a, \tau b)/(a,b) \in \bar{R} , \ \tau \in R_M^{(n)}\}$ *)*
Where $\bar{R}$ stands for the equivalence generated by R.

3 Examples of Polypodes

We are going to exhibit some remarkable examples of polypodes.

Example 1. Let Σ be an ordinary alphabet and $X_n = \{x_1, ..., x_n\}$ our set of variables. Word substitution is a polypodic operation converting $(\Sigma \cup X_n)^*$ into an $n - polypode$. Further the set $P((\Sigma \cup X_n)^*)$ with OI language substitution becomes an $n - polypode$ as well.

Example 2. Take the free polypode $T_\Gamma(X_n)$. Then the set $P(T_\Gamma(X_n))$ with OI forest substitution becomes an $n - polypode$

The fundamental functions on trees :
$yield : T_\Gamma(X_n) \to X_n^*$
$branch : T_\Gamma(X_n) \to P(T_{b(\Gamma)}(X_n))$
are morphisms of polypodes and ordinary tree homomorphisms is nothing but morphisms between free polypodes.

Example 3. Let E be an ordinary alphabet and $X_n = \{x_1, ..., x_n\}$. $Graph(EX_n)$ stands for the set of all finite , directed graphs whose edges are labelled over the set $E \cup X_n$ and we assume they have a distinguished edge (with distinct ends) as a root:

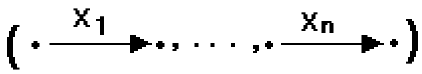
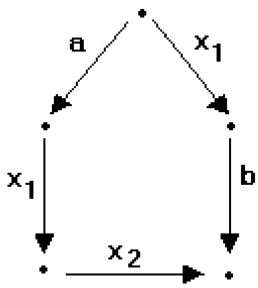


Figure 1:



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Edge substitution structures $Graph(EX_n)$ into a polypode whose unit is the n-tuple:

Example Polypodes of functions *Let* A *be a non empty set and* $[A^n, A]$ *the set of all functions from* A^n *to* A. *For* $f, g_1, ..., g_n \in [A^n, A]$, *the composition*

$f \circ (g_1, ..., g_n) : A^n \to A$

is given by

$f \circ (g_1, ..., g_n)(q_1, ..., q_n) = f(g_1(q_1, ..., q_n), ..., g_n(q_1, ..., q_n))$

for all $(q_1, ..., q_n) \in A^n$.

An easy argument shows that $[A^n, A\}$ *with the above operation is an* $n-$ *polypode*; *its unit is composed by the* n *projections :*

$pr_i : A^n \to A$, $pr_i(q_1, ..., q_n) = q_i$ $(1 \leq i \leq n)$

Subpolypodes of $[A^n, A]$ *are referred by Cohn as clones (cf.[Cohn 65]).*

The following are subpolypodes of $[N^n, N]$, N the set of natural numbers :

a. The set of polynomial functions, that is functions $f : N^n \to N$ given by rules of the form

$f(x_1, ..., x_n) = \underset{i_1,...,i_n}{\Sigma} a_{i_1...i_n} x_1^{i_1}...x_n^{i_n}$ (finite sum).

b. The set of Affine functions that is functions $f : N^n \to N$ given by rules of the form

$f(x_1, ..., x_n) = a_1x_1 + \cdots + a_nx_n + b.$ $a_i, b \in N$

4 Polypodic Invariants

Syntax can be defined in any polypode by means of $\Delta_N^{(n)}$ -action.Precisely, let $L \subseteq M$ and $m \in M$. The **context** of m according L is the set of all trees $\tau \in \Delta_M^{(n)}$ such that $\tau m \in L$:

$cont_L(m) = \{\tau \in \Delta_M^{(n)} / \tau m \in L\}$

Merging elements with the same context we define a polypodic congruence $\equiv_L$ on M called **syntactical**. The quotient

$M_L = M / \equiv_L$

is the **syntactical n-polypode of L.**

$L \subseteq M$ is said to be **globally recognizable** if there exists a polypode morphism $h : M \to N$,

N finite, so that

$L = h^{-1}(P)$, for some $P \subseteq N$

Theorem 4. *Next conditions are equivalent:*

i)$L \subseteq M$ is globally recognizable

ii)L is saturated by a finite index polypodic congruence

iii)The syntactical polypodic congruence $\equiv_L$ has finite index

iv)The syntactical polypode M_L is finite

v)$Card\{\tau^{-1}L / \tau \in \Delta_M^{(n)}\} < \infty$ where $\tau^{-1}L = \{m \in M / \tau m \in L\}$

Remark. In the case of monoids $(1-polypodes)$ the above theorem is translated into known facts (cf [Eilenberg 74])

Remark. In the case $M = T_\Gamma(X_n)$ one can add one more equivalent condition in the string i)-v) of the previous theorem:

vi)$L \subseteq T_\Gamma(X_n)$ is recognizable in the usual sense.

Therefore items i)-v) above give new forest recognizability characterizations.

Remark. Graph recognizability presents a special interest and will be the subject of a forth coming work.

Right derivatives are defined with respect to $P_M^{(n)}$-action .More precisely for $L \subseteq M$ and $m \in M$ we put

$Lm^{-1} = \{\tau \in P_M^{(n)} / \tau m \in L\}$

Once again , merging elements with the same right derivative we define a bottom congruence $\sim_L$ which saturates L and actually it is the finest bottom congruence with this property.

Theorem 5. *Next conditions are equivalent:*

i)$L \subseteq M$ is globally recognizable.

ii)L is saturated by a finite index bottom congruence

iii)The bottom congruence $\sim_L$ has finite index

iv)$card\{Lm^{-1} / m \in M\} < \infty$

Polypodic invariants are used to establish Eilenberg's theorem on varieties into the framework of trees (cf.[Bozapalidis2 99]).

5 Local Recognizability

Locally recognizable subsets in polypodes we are going to introduce , play a role corresponding to rational subsets in $\Gamma - a\lg ebras$ .Let M be an $n - polypode$. A subset $L \subseteq M$ is termed **locally recognizable** if there exist a finite $n - ranked$ alphabet Γ and a polypode morphism

$a : T_\Gamma(X_n) \rightarrow M$

so that $L = a(R)$ for some recognizable forest $R \subseteq T_\Gamma(X_n)$.

In the case M is a finitely generated polypode the class of its globally recognizable subsets is included into that of locally recognizable subsets.

$GREC(M) \subseteq LREC(M)$

This inclusion is proper when we are dealing with trees, $M = T_\Gamma(X_n)$.

How about machines computing locally recognizable subsets?

A **local automaton** is a 4-tuple

$A = (\Pi, Q, (d_1, ..., d_n), F)$

formed by a finite subset Π of an $n - polypode$ M a finite set Q of states , $(d_1, ..., d_n)$ of **final** states. The moves of A are described by a $\Pi-$indexed family of functions

$a_\pi : Q^n \rightarrow Q \quad , \quad \pi \in \Pi$

These functions collectively define a single function

$a : \Pi \rightarrow [Q^n, Q]$

which is uniquely extended into a polypode morphism

$\bar{a} : T_\Pi(X_n) \rightarrow [Q^n, Q]$

Now , the behavior of A is obtained into the following three steps: we first consider the set $\breve{F}$ of all functions $f : Q^n \rightarrow Q$ sending the initial $n - tuple$ $(d_1, ..., d_n)$ into a final state

$\breve{F} = \{f \in [Q^n, Q] / f(d_1, ..., d_n) \in F\}$

we then get all trees $t \in T_\Pi(X_n)$ that are interpreted by $\hat{a}$ into such a function

$\hat{a}^{-1}(\breve{F}) = \{t \in T_\Pi(X_n) / \hat{a}(t) \in \breve{F}\}$

and finally we complete all such trees into M :

$|A| = val(\hat{a}^{-1}(\breve{F}))$

where $val : T_\Pi(X_n) \rightarrow M$ is the canonical polypode morphism extending the inclusion $\Pi \subseteq M$

Proposition 6. *$L \subseteq M$ is locally recognizable if and only if $L = |A|$ for some local automaton A*

Example 4. Take $M = X_n^*$ and consider the local automaton

$A = (\{x_1^2\}, \{q_0, q_+\}, (q_0, ..., q_n), \{q_+\})$
with move function
$a_{x_1^2} : Q^n \to Q$
given by
$a_{x_1^2}(q_0, ..., q_0) = q_+ = a_{x_1^2}(q_1, ..., q_+)$
It is easily seen that
$|A| = \{x_1^{2^n} / n \geq 0\}$
which is a non recognizable language

Theorem 7. *If $F \subseteq T_\Gamma(X_n)$ is a locally recognizable forest then $yield(F) \subseteq X_n^*$ is a locally recognizable language and $branch(F)$ is a recognizable monadic forest*

Next theorem gives us information about stability of locally recognizable sets under the action of polypode morphisms and its inverse

Theorem 8. *Let $h : M \to M'$ a polypode morphism*
i)If $L \subseteq M$ is locally recognizable then so is $h(M) \subseteq M'$.
ii)If $L' \subseteq M'$ is locally recognizable and h has finite fibers (i.e. for each $m' \in M'$, $h^{-1}(m')$ is finite), then $h^{-1}(L')$ is locally recognizable as well.

We immediately deduce

Corollary 9. *Forest local recognizability is preserved by inverse non deleting tree homomorphisms*

Furthermore

Theorem 10. *Assume that L,$L_1, ..., L_n$ are locally recognizable subsets of an $n-$polypode M.Then*
$L[L_1, ..., L_n]$ and $pol(L)$ are locally recognizable too

Let us denote by $PRAT(M)$ the least class containing the finite subsets of the polypode M and closed under the operations of "polypodic product"
$(L, L_1, ..., L_n) \mapsto L[L_1, ..., L_n]$
and "polypodic star"
$L \mapsto pol(L)$
Then
$LREC(M) \subseteq PRAT(M)$
A last closure result

Proposition 11. *In any polypode M the intersection of a globally recognizable subset with a locally recognizable subset is a locally recognizable subset.*

Local recognizability in polypodes of functions presents great interest.

Let A be a non empty set and take $M = [A^n, A]$ .To every function $f : A^n \to A$ we associate an $n - ranked$ symbol $\overset{\frown}{f}$.

According to proposition 5.1 a set of functions $L \subseteq [A^n, A]$ is locally recognizable if there exist a finite set of functions $\Pi \subseteq [A^n, A]$ and a recognizable forest $R \subseteq T_{\hat{\Pi}}(X_n)$, where $\hat{\Pi} = \{\hat{\pi} \,/ \pi \in \Pi\}$ so that

$L = a(R)$

$a : T_{\hat{\Pi}}(X_n) \to [A^n, A]$ denoting the canonical polypode morphism extending the interpretation of each symbol $\hat{\pi}$ by the function π.

Example 5. We are going to show that the set of all polynomial functions with n variables is a locally recognizable subset of $[IN^n, IN]$ IN the set of natural numbers. For simplicity sake we treat the case $n = 2$. Recall that a function $f : IN^2 \to IN$ is said **polynomial** if it can be written as a finite sum as follows:

$f(x, y) = \sum_{\kappa,\lambda} a_{\kappa,\lambda} x^\kappa y^\lambda$

with the coefficients $a_{\kappa,\lambda} \in IN$

Introduce the following functions from IN^2 to IN :

$f_0(x, y) = 0$ $\quad\quad f_1(x, y) = 1$

$f_x(x, y) = x$ $\quad\quad f_y(x, y) = y$

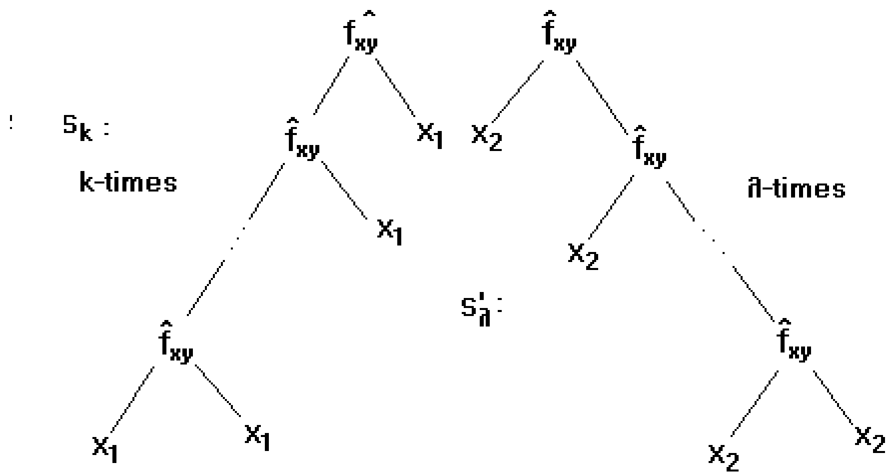
$f_{x+1}(x, y) = x + 1$ $\quad\quad f_{y+1}(x, y) = y + 1$

$f_{x+y}(x, y) = x + y$ $\quad\quad f_{xy}(x, y) = xy$

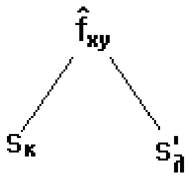
and the $2 - ranked$ alphabet

$\Gamma = \{\hat{f}_0, \hat{f}_x, \hat{f}_y, \hat{f}_{x+1}, \hat{f}_{y+1}, \hat{f}_{x+y}, \hat{f}_{xy}\}$.

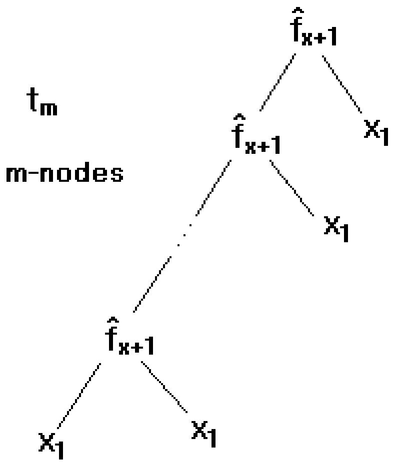
The trees



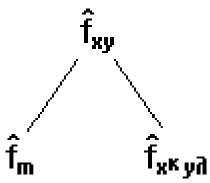
are interpreted by the functions
$f_{x^k}(x,y) = x^k \quad f_{y^\lambda}(x,y) = y^\lambda$
Thus



is interpreted by the functions
$f_{x^\kappa y^\lambda}(x,y) = x^\kappa y^\lambda \qquad \kappa, \lambda \geq 2.$
On the other hand the tree



represents the constant at m function $f_m(x,y) = m$ for all $x, y \in IN$. Therefore



represents the monomial functions $f_{mx^{\kappa}y^{\lambda}}(x,y) = mx^{\kappa}y^{\lambda}$

Finally a finite sum of such terms is obtained using the symbol $\hat{f}_{x+y}$.

The whole forest $T_{\Gamma}(X_n)$ is consequently projected onto $Pnom[IN^2, IN]$.

Removing the function f_{xy} from the previous list we get a $2-ranked$ alphabet Γ' all trees over which realize the set of affine functions

$Aff[IN^2, IN] = \{f(x,y) = \alpha x + \beta y + \gamma / \alpha, \beta, \gamma \in IN\}$

Conclusion:There is a finite machine computing all polynomial functions another computing all affine functions etc.

6 Other recognizability notion

a. Top recognizability.

Let M be an $n-polypode$. The set M^n can be structured into a monoid with multiplication:

$(m_1, ..., m_n)(m_1', ..., m_n') = (m_1[m_1', ..., m_n'], ..., m_n[m_1', ..., m_n'])$

and unit $(e_1, ..., e_n)$

Now let $L \subseteq M$.

Theorem 12. *Next conditions are equivalent:*

i)L is saturated by a finite index top congruence

ii)$L \times \cdots L$ is a recognizable subset of the monoid M^n

iii)$card\{\tau^{-1}L / \tau \in R_M^{(n)}\} < \infty$

iv)$card\{Lm^{-} / m \in M\} < \infty$

where $Lm^{-} = \{\tau \in R_M^{(n)} / \tau m \in L\}$.

L is said to be **top recognizable** whenever it satisfies any of the above equivalent conditions.

Call $L \subseteq M$ **projectively recognizable** (resp. **rational**) if there exists a recognizable (resp. rational) subset $A \subseteq M^n$ such that $L = pr_i(A)$ for some index i $(1 \leq i \leq n)$.

We have next string of inclusions:

$GREC(M) \subseteq TREC(M) \subseteq P_rREC(M)$

where $TREC(M)$ and $P_rREC(M)$ stand for the classes of all top and projectively recognizable subsets of M.

b. Partial recognizability

To each $n - polypode$ M with unit $e = (e_1, ..., e_n)$ n multiplications are associated as follows:

$m_k m' = m[e_1, ..., e_{k-1}, m, e_{k+1}, ..., e_n] \quad (1 \leq k \leq n)$.

for all $m, m' \in M$. They are associative and admit the elements $e_1, ..., e_n$ respectively as units.

These multiplications correspond to substitution to a specified variable x_k $(1 \leq k \leq n)$

Therefore the polypode M becomes a monoid into n different ways.

We call such structures multimonoids. Precisely an **n-monoid** is nothing but a set A with n structures of monoid.

A subset L of an $n - polypode\ M$ is said to be **partially recognizable** if there exist a finite $n - monoid\ A$ and an $n - monoid$ morphism $g : M \to A$ so that $L = g^{-1}(P)$, for some $P \subseteq A$.

It is easily seen that any globally recognizable subset of M is partially recognizable and the family

$ParREC(M)$ of such subsets is closed under boolean operations and inverse polypode morphism. Derivative characterizations of this kind of recognition can also be obtained

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